Completion of Dominant K-Theory

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Abstract

Nitu Kitchloo generalized equivariant K-theory to include non-compact Kac-Moody groups, calling the new theory Dominant K-theory. For a non-compact Kac-Moody group there are no non-trivial finite dimensional dominant representations, so there is no notion of a augmentation ideal, and the spaces we can work with have to have compact isotropy groups. To resolve these we complete locally, at the compact subgroups. We show that there is a 1 dimensional representation in the dominant representation ring such that when inverted we recover the regular representation ring. This shows that if H is a compact subgroup of a Kac-Moody group $\mathcal{K}(A)$, the completion of the Dominant K-theory of a H-space X is identical to the equivariant K-theory completed at the augmentation ideal. This is the local information. To glue this together we find a new spectrum whose cohomology theory is isomorphic to $K^*(X \times_{\mathcal{K}} E\mathcal{K})$. This enables us to use compute $K^*(X \times_{\mathcal{K}} E\mathcal{K})$ using a skeletal filtration as we now know the E_1 page of this spectral sequence is formed out of known algebras.
READERS: Professor Nitu Kitchloo (Advisor), Jack Morava, Steve Wilson, Petar Maksimovic, and James Spicer

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Introduction

K-theory was originally defined by Michael Atiyah and Friedrich Hirzebruch in 1959 using work on sheaves by Alexander Grothendieck. Atyiah and Hirzebruch used Grothendieck's construction to vector bundles over topological spaces instead of sheaves over an algebraic variety. A vector bundle, V, over topological space, X, is a collection of vector spaces attached at each point that varies continuously. To get the K-theory of a space X, we consider the category of all vector bundles over X which has an algebraic operation of direct sum. Two bundles can be added together by taking the cartisian product of the vector spaces over each point of X. Then, we formally invert the direct sum operation in the following way. Instead of considering just bundles, V we consider pairs of bundles where one is considered to be "negative", $V - W$. Then, we identify all virtual bundles of the form $V - V$ with 0. This process gives us the K-theory of X, $K^*(X)$. K-theory was one of the main inspirations for generalized cohomology theories which is at the heart of modern algebraic topology.

Later on Grahm Segal produced a variant of K-theory where he considered spaces that have a group action $G \times X \to X$, and made the vector bundles over these G-spaces respect this action. This produced a much richer equivariant K-theory, $K^*_G(X)$. For equivariant K-theory Atiyah and Segal proved the following theorem:

Theorem 1.0.1 (Atiyah-Segal Completion Theorem [2]). *Let G be a compact Lie group. The projection $X \times EG \to X$ induces an isomorphism:*

$$K^*_G(X)^{\wedge}_{I_G} \cong K^*_G(X \times EG)$$

Where I_G is the augmentation ideal of the representation ring of G.

The augmentation ideal I_G, is the kernel of the map from the representation ring of G, R_G, to $\mathbb{Z}$ that sends a representation to its dimension. This theorem provides a simpler way to compute the K-theory of the classifying space of G-bundles, $K^*(BG)$; just take X to be a point and work with the simpler algebraic completion. However the group G needs to be compact for technical reasons.

In 2009 Nitu Kitchloo generalized equivariant K-theory of spaces with a Lie group action to dominant K-theory where the groups were now allowed to be infinite dimensional Kac-Moody groups [11]. Compact Lie groups are classified by a Cartan matrix A. A matrix is a Cartan matrix if it satisfies four conditions. From such a matrix we can construct a Lie group. However, one of the conditions only exists to ensure that the Lie groups is finite dimensional. If we relax this one condition the matrix is called a generalized Cartan matrix, and we can still construct a group $\mathcal{K}(A)$ from a given generalized Cartan matrices. These could be infinite dimensional and thus non-compact. For this work $\mathcal{K}(A)$ will be an arbitrary Kac-Moody group, usually non-compact and abbreviated $\mathcal{K}$. The notation used for this dominant K-theory on a space X is $\mathbb{K}^*_{\mathcal{K}}(X)$.

In Kitchloo's generalization there are two important restrictions. We must restrict ourselves to $\mathcal{K}(A)$-spaces whose isotropy groups are all compact, called proper spaces. Secondly, the spectrum representing dominant K-theory, $\mathbb{KU}$ is built out of only a subset of all irreducible representations of $\mathcal{K}$ that are called dominant. This enables us to use the well understood dominant representation theory of Kac-Moody groups that strongly resembles the classical representation theory of Lie groups. These restrictions only matter when $\mathcal{K}$ is non-compact. When $\mathcal{K}$ is compact, dominant K-theory is equivariant to classical equivariant K-theory as all representations are dominant and all $\mathcal{K}$-spaces are proper. The aim of this work is to generalize the Atiyah-Segal completion theorem of equivariant K-theory to dominant K-theory.

For non-compact Kac-Moody groups there are no non-trivial finite dimensional dominant representations. This means that there is no dimension map on the representations of $\mathcal{K}(A)$, and no augmentation ideal to complete at. However, one could ask if there could be a completion with respect to a different ideal that gives a similar result. We will show that such a completion is impossible for dominant K-theory.

Our notion of completion will be of a more geometric flavor. We will use the long running theme in mathematics of gluing together local information to form the global information. Each proper $\mathcal{K}$-space M can be given a $\mathcal{K}$-CW structure. In other words we can build M up from orbit spaces of the form $\mathcal{K} \wedge_H S^n$. We know that $\mathbb{K}^*_{\mathcal{K}}(\mathcal{K} \wedge_H S^n)$ is a subring of the classical representation ring R_H, called the dominant representation ring DR_H. At each orbit, i.e. $\mathcal{K} \wedge_H S^n$ we show that this can be completed, and it's completion is isomorphic to the completion of R_H.

Theorem 3. *There is an isomorphism*

$$(DR_H)^\wedge_{I\cap DR_H} \cong R_H{}^\wedge_I,$$

where I is the augmentation ideal of R_H.

To show that this local information glues together in a well-defined way (that it does not depend on out choice of $\mathcal{K}$-CW complex for M), we'll find a new spectrum to represent completion and a map to it from the spectrum for dominant K-theory. Specifically we show the following result:

Theorem 3.2.1. *There is a space $\widehat{\mathcal{F}(\mathcal{H})}$ and a $\mathcal{K}$-equivariant map $c : \mathcal{F}(\mathcal{H}) \to \widehat{\mathcal{F}(\mathcal{H})}$, such that*

$$\mathbb{K}_{\mathcal{K}}(M) \xrightarrow{c} [M, \widehat{\mathcal{F}(\mathcal{H})}]_{\mathcal{K}} \cong K(M \times_{\mathcal{K}} E\mathcal{K})$$

and $\widehat{\mathcal{F}(\mathcal{H})}$ is non-equivariantly homotopic to $\mathcal{F}(\mathcal{H})$. Furthermore, the map c lifts to a map of spectra $c : \mathbb{KU} \to \widehat{\mathbb{KU}}$ and fits into the following diagram, where the bottom sequence is short exact:

$$\begin{tikzcd}
 & & & \mathbb{K}^*_{\mathcal{K}}(M) \arrow[dl, "c"'] \arrow[d] & \\
0 \arrow[r] & \varprojlim^1_n \mathbb{K}^*_{\mathcal{K}}(M \times \mathcal{K}^{*n}) \arrow[r] & K^*(M \times_{\mathcal{K}} E\mathcal{K}) \arrow[r] & \varprojlim_n \mathbb{K}^*_{\mathcal{K}}(M \times \mathcal{K}^{*n}) \arrow[r] & 0
\end{tikzcd}$$

The space $\mathcal{F}(\mathcal{H})$ is the space of Fredholm operator and is the space that $\mathbb{KU}$ is built out of. Using the spectrum $\widehat{\mathbb{KU}}$ with a $\mathcal{K}$-CW structure gives us a spectral sequence that computes the global object from the local data,

$$E_1^{s,*} = \bigoplus_i (R_{H_{i,s}})^\wedge_I[\beta^\pm] \Rightarrow \mathbb{K}^*_{\mathcal{K}}(M \times_{\mathcal{K}} E\mathcal{K}), \tag{1.0.1}$$

where $M_s/M_{s-1} = \bigvee_i \mathcal{K} \wedge_{H_i} S^s$ and $\{M_s\}_{s=0}^\infty$ is a $\mathcal{K}$-CW filtration of M.

2

Background

2.1 Background: Equivariant K-theory

This section is a short account of equivariant K-theory and leads up to the Atiyah-Segal completion theorem. This document assumes that the reader is familiar with K-theory. As a reference we give the excellent book by Atiyah [3] for details on K-theory. We also assume that the reader understands the representation theory of Lie groups. A good reference for this is Bröcker and tom Dieck's book [5].

2.1.1 Definition of Equivariant K-theory

The original definition of K-theory of a space X is the Grothendiek construction on the category of isomorphism classes of complex vector bundles over X under direct sum. This construction is not used to define dominant K-theory, so we will give a quick overview of how to use Fredholm operators to define a representing object for equivariant K-theory.

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. We choose a Hilbert space $\mathcal{H}$ called a *universe*.

$$\mathcal{H} = \bigoplus_{\lambda} \mathbb{C}^{\infty} \otimes V_{\lambda},$$

where the sum is taken over all irreducible representations of G, V_{λ}, indexed by their heighest weights, λ. To each V_{λ} we can associate a G-invariant inner product and the inner product on $\mathcal{H}$ is constructed from these. $\mathcal{H}$ is maximal as for any representation we have $\mathcal{H} \oplus V \cong \mathcal{H}$. From the theorem of Peter and Weyl we know $\mathcal{H} \cong L^2(G)$ for a compact group G, where $L^2(G)$ are L^2 functions on G. A linear operator $f : \mathcal{H} \to \mathcal{H}$ is Fredholm if its kernel and cokernel are finite dimensional. Let $\mathcal{F}(\mathcal{H})$ be the set of Fredholm operators on $\mathcal{H}$. To each Fredholm operator we can associate an index:

$$\mathrm{index}(f) = \dim(\ker(f)) - \dim(\mathrm{coker}(f)).$$

However, as the Fredholm operators are meant to encapsulate information about the vector bundles and not just vector spaces, we will instead use the topological index:

$$\mathrm{index}(f) = \ker(f) - \mathrm{coker}(f).$$

This index maps to the category of isomorphism classes of complex vector spaces with direct sum formally inverted which is isomorphic to $\mathbb{Z}$. With more work that is omitted, this index can be extended to topological spaces and produces the isomorphism

$$[X, \mathcal{F}(\mathcal{H})] \xrightarrow{\mathrm{index}} K^0(X).$$

From this we can see that $\mathcal{F}(\mathcal{H})$ is the classifying space of K-theory. This is no different from the standard index as we have not imposed any action of G on $\mathcal{F}(\mathcal{H})$. We can give $\mathcal{F}(\mathcal{H})$ a G-action by

$$(g \cdot f)(x) = g(f(g^{-1}x)).$$

We would like this to be a representing object for equivariant K-theory, but the action is not continuous in the standard norm topology. One can fix this in a few ways. Atiyah and Segal restrict $\mathcal{F}(\mathcal{H})$ to a subspace

$$\mathcal{F}_{G-\mathrm{cts}}(\mathcal{H}) = \{u \in \mathcal{F}(\mathcal{H}) | g \mapsto gug^{-1} \text{is continuous}\},$$

with a similar condition on $\mathcal{F}_{G-\mathrm{cts}}(\mathcal{H})$. Here, the group action is continuous and $\mathcal{F}_{G-\mathrm{cts}}(\mathcal{H})$ does form a representing object for equivariant K-theory. Please see the appendix of "Twisted K-theory" for details [1]. We will refer to $\mathcal{F}_{G-\mathrm{cts}}(\mathcal{H})$ as $\mathcal{F}(\mathcal{H})$ for the rest of the work as there is no need for the full space of Fredholm operators anywhere. We may then define equivariant K-theory for a G-space, X, to be the homotopy classes of G-equivariant maps $X \to \mathcal{F}(\mathcal{H})$ or $K^0_G = [X, \mathcal{F}(\mathcal{H})]_G$. As $\mathcal{H}$ is maximal, i.e. $\mathcal{H} = \mathcal{H} \otimes \mathcal{H}$, it satisfies Bott periodicity and we may define a naive (indexed over a trivial universe) G-spectrum where the even objects are $\mathbb{K}_{2n} = \mathcal{F}(\mathcal{H})$ and the odd are $\mathbb{K}_{2n+1} = \Omega\mathcal{F}(\mathcal{H})$.

So

$$K_G^{2n}(X) = [X, \mathcal{F}(\mathcal{H})]_G \quad \& \quad K_G^{2n+1}(X) = [X, \Omega\mathcal{F}(\mathcal{H})]_G.$$

This is a G-equivariant cohomology theory, K_G^*, on G-spaces. There are better constructions that produce genuine G-spectra, but as these do not apply to dominant K-theory we will not cover them.

2.1.2 Equivariant K-theory of a point

Arguably the most important space to ask about when dealing with a new cohomology theory is the point. In the case of equivariant K-theory this turns out to be the representation ring of the underlying group R_G. This is well know and easy to prove from the construction of equivariant K-theory using vector bundles [20]. The easy way to see this, for our definition that uses Fredholm operators, is to recall that $[pt, \mathcal{F}(\mathcal{H})]_G$ will be the path connected components of the G fixed points of the Fredholm operators, $\mathcal{F}(\mathcal{H})_G = \{f | g \cdot f = f\}$. The fixed points are the operators $\mathcal{H} \xrightarrow{f} \mathcal{H}$ that are G-equivariant. The kernel and cokernel of equivariant Fredholm operators are G-representations, so the index map's codomain is isomorphism classes of complex representations of G with direct sum formally inverted, also known as R_G. Therefore

$$K_G^*(pt) = R_G[\beta^{\pm 1}].$$

Similarly, we can show that for any G-space X with trivial action we have:

$$K_G^0(X) = K^0(X) \otimes R_G.$$

2.1.3 Equivariant K-theory of a free G-space

A large class of interesting examples of G-spaces have a free G-action. If X has a free G-action, then $K_G^*(X) = K^*(X/G)$. Again, this is easy to see when we use the bundle construction of equivariant K-theory [20], but we are using the Fredholm operator model. Later on, we prove that there is a map $K_G^*(X) \to K^*(X \times_G EG)$ (3.2.1). The proof that we present is for dominant K-theory (which will be defined later) but it applies equally well to equivariant K-theory. As X is a free G-space then $X \times_G EG$ is a trivial EG-bundle over X/G. Therefore as EG is contractible, $K_G^*(X) = K^*(X/G)$.

2.1.4 Equivariant K-theory of an orbit G/H

Now that we know the equivariant K-theory of G-spaces with either a free or trivial G-action we turn our attention to those whose isotropy groups are between the trivial subgroup and the full group. The simplest of which are the orbit spaces G/H, for some $H \subset G$. However, to compute $K^*_G(G/H)$ we will use a much stronger result as it is relevant later. If H is a subgroup of G then there is a functor $G \times_H \_ : H - \text{top} \to G - \text{top}$, where $H - \text{top}$ is the category of H-spaces and $G - \text{top}$ is the category of G-spaces. $G \times_H \_$ is a right adjoint to the forgetful functor $U : G - \text{top} \to H - \text{top}$. As we are working in the category of compact Lie groups, we have that every irreducible representation of H occurs as a subrepresentation of some representation of G. One way to see this is given a H representation there is a way to induce a G-representation. See [21] for details. This implies that $\mathcal{H}_H = U\mathcal{H}_G$, where $\mathcal{H}_H$ is the universe for H, and similarly for G and U is the forgetful functor. This lifts to the Fredholm operators, $\mathcal{F}(\mathcal{H}_H) \cong U\mathcal{F}(\mathcal{H}_G)$, and the spectra that define equivariant K-theory for the respective groups giving us:

Lemma 2.1.1. *Suppose X is a H space, $H \subset G$ then:*

$$K^*_H(X) = K^*_G(X \wedge_H G_+).$$

This gives us $K^*_G(G/H) = K^*_G(pt \wedge_H G_+) = K^*_H(pt) = R_H[\beta^{\pm 1}]$.

2.1.5 The Augmentation Ideal

The projection map $G \to pt$ induces the map $K^*_G(pt) \to K^*_G(G) \cong K^*(pt)$. This can be simplified to the dimension map $R_G \to \mathbb{Z}$, which sends each irreducible representation in R_G to its dimension in $\mathbb{Z}$. The augmentation ideal is the kernel of this map, thus we have a short exact sequence:

$$0 \to I_G \to R_G \to \mathbb{Z} \to 0.$$

Note that,

Lemma 2.1.2. *If G is a Lie group and T is a maximal torus of G and $i : T \to G$ is the inclusion then:*

$$I_G = i^*(I_T).$$

Proof. We know that the map $R_G \to R_T$ induced by inclusion is injective. We also know that $R_G \to R_T^W$ is an isomorphism where W is the Weil group associated to G and T. As R_T^W's augmentation ideal is clearly the pullback of the augmentation ideal of R_T we are done. □

We need the following result for the proof of the completion theorem. If we have a subgroup $H \subset G$ then we can always pull the augmentation ideal of H to the representation ring R_G, however it need not be the augmentation ideal of G.

Corollary 2.1.2.1. *If H is a subgroup of G such that H contains some maximal torus of G then $I_G = i^*(I_H)$ where i is the inclusion map of H into G.*

Proof. We know that $e \to H \xrightarrow{i} G$ induces a factorization of the dimension map $R_G \to R_H \to \mathbb{Z}$, so $i^*(I_H) \subset I_G$. Let T be maximal in both H and G. Then the map $R_G \to R_T$ factors through the map $R_H \to R_T$. By the previous lemma, both of the augmentation ideals for H and G are obtained by pulling the augmentation ideal of T back through these maps. So $i^*(I_H) = I_G$ as it has to be maximal. □

2.2 The Atiyah-Segal Completion Theorem

Theorem 2.2.1 (Atiyah-Segal Completion Theorem)**.** *Let G be a compact Lie group. The projection $X \times EG \to X$ induces an isomorphism:*

$$K_G^*(X)^\wedge_{I_G}(X) \cong K_G^*(X \times EG).$$

The proof follows a bootstrap style argument. We first deal with the case where G is a torus, then we use that to prove the result for $G = U(n)$ for some n. Now using the fact that every compact Lie group G has a faithful unitary representation $G \to U(n)$ we can finally show the complete result. Before we begin the proof, we review algebraic completion and Milnor's join model for EG.

2.2.1 Algebraic Completion

Let R be a ring and I an ideal in R. We may endow R with the "I-adic topology". For a subset $U \subset R$ such that $0 \in U$, U is open if $U \subset I^j$ for some j. A sequence $\{a_j\}_{j=0}^\infty$ is Cauchy in this topology if for all r there is an N such that for all $n, m > N$ we have $a_n - a_m \in I^r$. The completion, $R_I^\wedge$ is topologically defined to be the space of all the Cauchy sequences modulo the following relation. Let $\{a_j\}_{j=0}^\infty$ and $\{b_j\}_{j=0}^\infty$ be two Cauchy sequences, we identify them if for all r there is an N such that for all $n > N$ we have $a_n - b_n \in I^r$. In other words $a_n + I^r = b_n + I^r$. So we can think of an Cauchy sequence as an element of $\bigoplus R/I^n$ such that $a_n + I^n = a_{n+1} + I^n$ (also known as coherent).

This motivates the following definition of the I-adic completion of R:

$$R_I^\wedge = \varprojlim R/I^n.$$

Let us complete $R[x]$ at the ideal $I = \langle x \rangle$. The easiest way to think of this is via coherent sequences in $\bigoplus R[x]/I^r$. Take $a_r + I^r \in R[x]/I^r$, it can be written $\sum_{i<r} b_{i,r} x^i + I^r$. This has to map to $a_{r-1} + I_{r-1}$, so:

$$\sum_{i<r} b_{i,r} x^i + I^{r-1} = \sum_{i<r-1} b_{i,r-1} x^i + I_{r-1}.$$

so $b_{i,r} = b_{i,r-1}$ for $i < r-1$. Let $R[[x]] \to R[x]/I^n$ be

$$\sum_{i=1}^{\infty} c_i x^i \mapsto \sum_{i=1}^{n-1} c_i x^i + I^n.$$

This clearly extends to a map $R[[x]] \to R[x]_I^\wedge$. It is easy to see that there is an inverse and $R[[x]] \cong R[x]_I^\wedge$. The same argument works for $R[x_1, \ldots, x_n]$ with the ideal $I = \langle x_1, \ldots, x_n \rangle$, giving us $R[[x_1, \ldots, x_n]] \cong R[x_1, \ldots x_n]_I^\wedge$. This computation allows us to make use of the following theorem from [7].

Theorem 2.2.2. *Let R be a ring and I be an ideal generated by the elements $a_1, \ldots a_n$. Then*

$$R_I^\wedge \cong R[[x_1, \ldots x_n]]/\langle x_1 - a_1, \ldots, x_n - a_n \rangle.$$

The classic example of completion is the p-adics, $\mathbb{Z}_p^\wedge = \mathbb{Z}_{p\mathbb{Z}}^\wedge$. From Theorem 2.2.2 they should be thought of as power series in p.

Another use of Theorem 2.2.2 is the computation of the completion at the augmentation ideal of the representation ring associated to the rotation group. Let $\mathbb{T}$ be the rotation group. From basic representation theory $R_\mathbb{T} = \mathbb{Z}[x^{-1}, x]$. The augmentation ideal is $I = \langle x - 1 \rangle$. So, from the theorem we have:

$$R_{\mathbb{T}}{}_I^\wedge = \mathbb{Z}[x^{-1}, x][[y]]/\langle y - (x-1) \rangle.$$

Let $t = x - 1$ then,

$$R_{\mathbb{T}}{}_I^\wedge = \mathbb{Z}[(t+1)^{-1}, t][[y]]/\langle y - t \rangle = \mathbb{Z}[[t]]. \tag{2.2.1}$$

We can drop $(t+1)^{-1}$ from the last term as it is invertible in $\mathbb{Z}[[t]]$.

2.2.2 Milnor's Model for EG

Any space that is contractible and has a free G action is a model for EG, the total space of the universal principal bundle, but the proof of the completion theorem uses is Milnor's join model. His original paper has more details, see [16]. Given any two spaces X and Y we may produce a new space called the join of X and Y:

$$X * Y = \frac{X \times [0,1] \times Y}{\sim},$$

where $(x, 0, y_1) \sim (x, 0, y_2)$, for all $y_i \in Y$ and $(x_1, 1, y) \sim (x_2, 1, y)$, for all $x_i \in X$. Notice that the map $X \hookrightarrow X * Y$ is null homotopic. To see this, pick a $y_0 \in Y$. We may factor the inclusion $X \to X * Y$ by $X * y_0 \cong \mathrm{Cone}(X)$, which is contactable. We may iterate this join to define the join of n spaces. An iterated n-join of a single space X^{*n} is n-connected, and the limit of the iterated joins is aspherical.

Now, let G be a compact Lie group. We may write an element of $G^{*(k+1)}$ as $(g_0, t_1, g_1, \ldots, t_k, g_{k+1})$ and let the G action be the obvious diagonal action. When we take the direct limit $\varinjlim G^{*n}$, we see that it is aspherical and has a free G-action, so it is a model for EG [17]. The nice part of this construction, is that we can cover G^{*n} with n contractible open sets.

2.2.3 Atiyah and Segal's Completion Theorem

A priori there is no connection between the completion of equivariant K-theory at the augmentation ideal of G and $K^*_G(X \times EG)$. To obtain this map we have to connect the pro-groups associated to the two sides, $\{K^*_G(X)/I^n_G K^*_G(X)\}$ and $\{K^*_G(X \times G^{*n})\}$. A priori we do not know that $\varprojlim\{K^*_G(X \times G^{*n})\}$ is $K^*_G(X \times EG)$, there could be a $\varprojlim^1$ term, but in the course of the proof we will see that the system is Mittag-Leffer and this $\varprojlim^1$ is 0.

As G^{*n}/G is covered by n contractible sets, the map $K^*_G(pt) \to K^*_G(G^{*n}) = K^*(G^{*n}/G)$ factors through $R_G \to R_G/I^n_G$. For a G-space X we can use this factorization and the naturality of the external product to factor $K^*_G(X) \to K^*_G(X \times G^{*n})$ through $K^*_G(X)/I^n_G K^*_G(X)$ giving us the map

$$K^*_G(X)/I^n_G K^*_G(X) \xrightarrow{\alpha_n} K^*_G(X \times G^{*n})$$

for each n. In the limit we get the map $K^*_G(X)^\wedge_{I_G} \to K^*(X \times_G EG)$ and the main theorem from [2] is that this map is an isomorphism. As a corollary to this we get that $K^*_G(X \times G^{*n})$ satisfies the Mittag-Leffler condition.

The proof produces maps β_n such that there exists a k and the following commutes:

$$\begin{tikzcd}
K_G^*(X)/I_G^{n+k}K_G^*(X) \arrow[r] \arrow[d] & K_G^*(X \times G^{*(n+k)}) \arrow[d] \arrow[ddl, "\beta_n"] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
K_G^*(X)/I_G^n K_G^*(X) \arrow[r] & K_G^*(X \times G^{*n})
\end{tikzcd}$$

We first produce these maps for the circle geoup then bootstrap up from the circle group to tori to the unitary group to the arbitrary one.

Step 1: The case of the circle group.

Lemma 2.2.3. *Let G be a compact Lie group and, X a compact G-space such that $K_G^*(X)$ is finite over R_G. Let $\theta : G \to T$ be a homomorphism by which G acts on ET then the homomorphisms*

$$\alpha : K_G^*(X)/I_T^n K_G^*(X) \to K_G^*(X \times T^{*n})$$

is an isomorphism of pro-rings.

This is the only part of the bootstrap where β_n is produced. One should think of θ as a character of G. The proof relies on the identification $T^{*n} \cong S^{2n-1}$. Recall that the augmentation ideal for T is generated by $\zeta = 1 - \rho$, where ρ is the standard representation of T. This can be used to generate a short exact sequence

$$0 \to K/\zeta^n K \to K_G^*(X \times S^{2n-1}) \to {}_{\zeta^n}K \to 0,$$

where $K = K_G^*(X)$ and ${}_{\zeta^n}K = \{x \in K | \zeta^n x = 0\}$. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K/I_T^{n+k}K \arrow[r] \arrow[d] & K_G^*(X \times S^{2n+2k-1}) \arrow[r] \arrow[d] \arrow[dl, "\beta_n"] & {}_{\zeta^{n+k}}K \arrow[r] \arrow[d, "\zeta^k"] & 0 \\
0 \arrow[r] & K/I_t^n K \arrow[r] & K_G^*(X \times S^{*n}) \arrow[r] & {}_{\zeta^n}K \arrow[r] & 0
\end{tikzcd}$$

As K is finitely generated over the Noetherian ring R_G the chain on the right hand side is eventually constant and there is a ζ^k that annihilates ${}_{\zeta^{n+k}}K$ for any k. A diagram chase proves the existence of β_n.

Step 2: When G is a torus T^n.

Here we induct on n to generalize the above theorem to tori. G will now have an action on

E_{T^n} induced by $\theta : G \to T^n$ and we will show $\alpha_n : K/I^m_{T^n}K \cong K^*_G(X \times ET^n)$. To do so we have to change our model for ET^n from Milnor's join model with system $\{(T^n)^{*m}\}$ to the equivalent $ET \times ET^{n-1}$ with cofinal system $\{T^{*p} \times (T^{n-1})^q\}$. This can be done in general, any system whose direct limit is EG and whose spaces are compact is isomorphic to Milnor's system. In this case the system $\{T^{*p} \times (T^{n-1})^q\}$ comes with maps

$$\alpha_{pq} : K/(a^p + b^q)K \to K^*_G(X \times T^{*p} \times (T^{*(n-1)})^q),$$

where a and b are the ideals in I_{T^n} generated by I_T and $I_{T^{(n-1)}}$ respectively. The pro-rings $\{K/I^m_{T^n}K\}$ and $\{K/(a^p + b^q)K\}$ are isomorphic, so we can just show that the latter is isomorphic to $K^*_G(X \times ET^n)$ instead. We have $K/(a^p + b^q)K \cong K \otimes_R R/a^p \otimes_R R/b^q$ and we can factor α_{pq} as

$$K \otimes_R R/a^p \otimes_R R/b^q \to K^*_G(X \times T^{*p}) \otimes_R R/b^q \to K^*_G(X \times T^{*p} \times (T^{*(n-1)})^q).$$

By the inductive assumption and the base case this produces an isomorphism of pro-objects [2].

Step 3: When G=U(n).

The inclusion of the maximal torus T, $j : T \to U$ induces a map $j^* : K^*_U(X) \to K^*_T(X)$. Atiyah and Segal use an earlier paper of Atiyah's [4] to produce a map $j_* : K^*_T(X) \to K^*_U(X)$. We obtain that in the case of unitary groups α_n is an isomorphism when η_n is:

$$\begin{tikzcd}
K^*_U(X)/I^{n+k}_U K^*_U(X) \arrow[r] \arrow[d, shift left, "j^*"] & K^*_U(X \times U^{*(n+k)}) \arrow[d, shift left, "j^*"] \\
K^*_T(X)/I^n_U K^*_T(X) \arrow[r] \arrow[u, shift left, "j_*"] & K^*_T(X \times U^{*n}) \arrow[u, shift left, "j_*"]
\end{tikzcd} \tag{2.2.2}$$

The topology induced by I_U on R_T is equivilant to the one induced by I_T. This follows since $I_U \subset I_T$ and there is a k such that $I^k_T \subset I_U$. As EU is also a universal space for T and U^{*n} is a cofinal system of compact T spaces. Therefore the bottom of the diagram 2.2.2 produces the isomorphism of pro-groups $\alpha : \{K/I^m_T K\} \cong \{K^*_T(X \times T^{*n})\}$.

Step 4: The general case.

Let G embed in U and let G act on U via this embedding, thus we have $K^*_U(X \times_G U) \cong K^*_G(X)$. As $(X \times_G U) \times U^{*n} = U \times_G (X \times U^{*n})$ we can complete K_G with respect to U instead of G. Therefore, we have that $\alpha_n : K^*_G(X)/I^m_U K^*_G(X) \cong K^*_G(X \times U^{*n})$ is an isomorphism of pro-objects. U^{*n} is cofinal with compact G-spaces we get the statement of the theorem, therefore $K^*_G(X \times U^{*n}) \cong K^*_G(X \times G^{*n})$.

We also have $K_G^*(X)/I_U^m K_G^*(X) \cong K_G^*(X)/I_G^m K_G^*(X)$ as the topologies induced by I_U and I_G coincide [21].

2.2.4 The Milnor Sequence

So we have shown that $K_G^*(X)_I^\wedge = \varprojlim_n K^*(X \times G^{*n})$, to relate it to $K^*(X \times_G EG)$ we need to use the Milnor $\varprojlim^1$ sequence. We have:

$$0 \to \varprojlim{}_n^1 K^*(X \times G^{*n}) \to K^*(X \times_G EG) \to \varprojlim_n K^*(X \times G^{*n}) \to 0.$$

We know that $\varprojlim_n K^*(X \times G^{*n})$ satisfies the Mittag-Leffer condition because the system $K^*(X \times G^{*n})$ is isomorphic to $K_G^*(X)/K_G^*(X)I_G^n$, which is eventually constant [2]. Therefore it has $\varprojlim_n^1 K^*(X \times G^{*n}) = 0$ and the completion theorem holds. In our generalization this will not hold, we will still have to consider the $\varprojlim^1$ in this exact sequence.

2.3 Kac-Moody Groups and Representation Theory

In the study of compact semi-simple Lie groups we classify their semi-simple Lie algebras by the Cartan matrix [5] [8].

Definition 1. *A Cartan matrix is a matrix such that*

1. $a_{ii} = 2$
2. $a_{ij} \leq 0$ *for* $i \neq j$
3. *If* $a_{ij} = 0$ *then* $a_{ji} = 0$
4. $a_{ij}a_{ji} < 4$

If we have such a matrix we can construct a compact Lie group. This construction first uses Serre's theorem to construct a Lie algebra. Then we exponentiate the resulting algebra to form a compact Lie group. However there is one condition we impose on Cartan matrices that ensures that the associated algebra is finite dimensional and serves no purpose in the construction of the algebra. A generalized Cartan matrix satisfies only the first 3 properties. The Serre relations still construct an algebra and we can study it as Victor Kac and Robert Moody did in the late sixties [18][9]. Exponentiating the group is no longer possible, but we can use an alternate method to construct the group associated to a Kac-Moody algebra. This is family of non-compact groups that behave like the compact the semi-simple Lie groups. Most of the following details come from Kumar's excellent "Kac-Moddy Groups and their Flag Variety" [14].

2.3.1 Kac-Moody Algebras

To construct a Kac-Moody algebra for a given $n \times n$ generalized Cartan matrix A we first construct what will be its Cartan subalgebra. A realization of A is a space $\mathfrak{h}$ of dimension $2n - \mathrm{Rank}(A)$ along with elements $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ and $\{\alpha_1^\vee \alpha_2^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}$ such that

$$\alpha_i(\alpha_j^\vee) = a_{ij}.$$

This realization is unique up to isomorphism. Let $I = \{1, 2, \ldots n\}$ be the indexing set for A. It should be thought of as indexing the roots and coroots of our realization. Also, the parabolic subgroups and other structures are built off of the subsets of the indexing set so we will need to refer to it often.

Generating the Algebra

Next, create the free Lie algebra over $\mathbb{C}$ on the symbols e_i and f_i for $i \in I$ and $\mathfrak{h}$ with the Serre relations:

1. $[\mathfrak{h}, \mathfrak{h}] = 0$,
2. $[h, e_i] = \alpha_i(h)e_i$ and $[h, f_i] = -\alpha_i(h)f_i$,
3. $[e_i, f_j] = \delta_{ij}\alpha_i^\vee$,
4. $\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0$ for $i \neq j$
5. $\mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0$ for $i \neq j$

The resulting algebra is called $\mathfrak{g}(A)$. Usually the Cartan matrix is clear from the context so we just say $\mathfrak{g}$. The resulting algebra has a triangular decomposition. Let $\mathfrak{n}$ be the subalgebra generated by the e_is, $\mathfrak{n}^-$ the subalgebra generated by the f_is, we have that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \oplus \mathfrak{n}^-$. We also have a root lattice $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$, with a subset $\Delta = \{\alpha \in Q | \mathfrak{g}_\alpha \neq 0\}$ called the roots. As before the positive roots, Δ^+ are those whose coefficients are positive. As in the finite dimensional case, $\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. The Borel subalgebra is $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$.

Useful Subalgebra

To each $J \subset I$ we may associate several subalgebras. Let

$$\Delta_J = \Delta \cap \bigoplus_{j \in J} \mathbb{Z}\alpha_j.$$

Then $\mathfrak{g}_J = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_Y} \mathfrak{g}_\alpha$. In this subalgebra there is a subalgebra isomorphic to $\mathfrak{g}(A_J)$, where A_J is obtained from A by removing all columns and rows not in J. We say that J is of finite type if A_J is a classical Cartan matrix. Note that $\{i\}$ is always of finite type as $A_{\{i\}} = [2]$. The parabolic subalgebra is $\mathfrak{p}_J = \mathfrak{g}_J \oplus \mathfrak{u}_J$, where $\mathfrak{u}_J = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_J^+}$. $\mathfrak{u}_J$ is also a subalgebra, called the nil-radical of $\mathfrak{p}_J$.

The Weyl Group

The Weyl group, W associated to $\mathfrak{g}$ is generated by $s_i : \mathfrak{h}^* \to \mathfrak{h}^*$ where

$$s_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i.$$

The Weyl group also acts on $\mathfrak{h}$

$$s_i(h) = h - \alpha_i(h)\alpha_i^\vee.$$

It is interesting to note is that not all roots are in the orbit of the α_is under the action of the Weyl group, unlike the classical case. We define a root α to be real if there exists a α_i and a $w \in W$ such that $w(\alpha_i) = \alpha$, and imaginary otherwise. The dimension of the weight space for imaginary roots may not be 1 and is in general unknown.

For an element $w \in W$ we say its length is the smallest l such that $w = s_{i_1}s_{i_2}\dots s_{i_l}$ and often denoted by $l(w)$. The interplay of the length of a Weyl group element and its action on the roots is powerful. The following lemma from [14] is very useful later on.

Lemma 2.3.1. *If $l(ws_i) \geq l(w)$ then $w(\alpha_i) \in \Delta^+$*

Proof. We will induct on $l(w)$. When $l(w) = 0$, $w = \mathrm{Id}$ and $\alpha_i \in \Delta^+$. If $l(w) > 0$ find a s_j such that $l(ws_j) = l(w) - 1$. We have that $s_i \neq s_j$, let W_J be the subgroup of W generated by s_i and s_j. Clearly for any $r \in W_J$ $l(r) \geq l_J(r)$. Chose an x such that $l(x) + l_J(x^-1w) = l(w)$ and $l(x)$ is at a minimum. It can be shown that $l(xs_i) \geq l(x)$ and $l(xs_j) \geq l(x)$. As $l(x) \leq l(w)$, by the inductive assumption, we have $x(\alpha_i), x(\alpha_j) \in \Delta^+$. Now consider $y = x^{-1}w$, as $l(ws_i) \geq l(w)$ we have $l(ys_i) \geq l(y)$. We can compute that $y(s_i) = p\alpha_i + q\alpha_j$ for $p, q \geq 0$. So,

$$w(\alpha_i) = x(y(\alpha_i)) = x(p\alpha_i + q\alpha_j) \in \Delta^+.$$

□

Knowing the action of an element of the Weyl group on the root elements is particularly useful when dealing with parabolic subalgebras. If we have a parabolic algebra $\mathfrak{g}_J$ its Weyl subgroup is the group generated by s_j for $j \in J$ and call it W_J. It is often useful to deal with the cosets of W/W_J through representatives so we define W'_J to be the collection of the shortest elements from each coset:

$$W'_J = \{w \in W | l(w) \leq l(ws_j) \text{for} j \in J\}.$$

There is also the Bruhat-Chevalley partial order. This can be thought of a a refinement of the length and is useful for defining an equivariant CW-structure on the Kac Moody group. For $v, w \in W$, we say $v \leq w$ if there exists $t_1, \ldots, t_p \in T := \{vs_iv^{-1} | v \in W, i \in I\}$ such that

1. $v = t_p \ldots t_1 w$ and
2. $l(t_j \ldots t_1 w) \leq l(t_{j-1} \ldots t_1 w)$ for all j.

Dominant Chamber

Let $\mathfrak{h}_{\mathbb{R}}$ be the real form of $\mathfrak{h}$, i.e. $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \otimes \mathbb{C}$, see Definition 1.4.1 in [14] for a complete description. We define the dominant Chamber of $\mathfrak{h}^*_{\mathbb{R}}$ to be $D_{\mathbb{R}} = \{\lambda | \lambda(\alpha_i^\vee) \geq 0\}$. In the compact case this is a a fundamental domain of the Weyl group's action on $\mathfrak{h}_{\mathbb{R}}$ but may not be in general. Usually we will refer to it just as D, suppressing the real form of the Cartan matrix. The Tits cone C is the orbit of D under the action of the Weyl group, $C = \mathbb{U}_{w \in W} wD_{\mathbb{R}}$. In the non-compact infinite dimensional case this is a strict subset of $\mathfrak{h}$, but the Tits cone is all of $\mathfrak{h}$ in the compact finite dimensional case. The following alternate characterization, also from Kumar [14], is very useful later on.

Proposition 1.

$$C = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} | \langle \lambda, \alpha^\vee \rangle < 0, \text{ \textit{only for finitely many} } \alpha \in \Delta^{\vee +}\}.$$

Again we relate a property of the Weyl group action to a property that can be verified by its action on a few elements of of $\Delta^{\vee +}$.

An example $\mathbb{T} \ltimes \tilde{L}SU(2)$.

For an illustrative example we can construct the algebra associated to the generalized Cartan matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$. This is the generalized Cartan matrix for the affine Lie algebra associated to $\mathfrak{sl}_2(\mathbb{C})$. We are considering only the algebra, not the group. The group is the affine loop group on $SL_2(\mathbb{C})$, $\mathbb{T} \ltimes \tilde{L}SU(2)$. We refer the reader to Pressley and Segal's "Loop Groups" for details [19]. The notation, and the choices involved, are borrowed from them and [12]. The affine Lie algebra can be expressed as $\mathbb{C}\langle h\rangle \oplus \mathbb{C}\langle d\rangle \oplus \mathfrak{su}_2[z, z^{-1}]$, with bracket given by $[f, g](z) = [f(z), g(z)] + k\mathrm{Res}\langle f, dg\rangle$ and $[d, f] = z\frac{df}{dz}$.

First, we construct the realization. Let $\mathfrak{h}$ have for a basis h, k, d and let $\mathfrak{h}^*$ have the basis where Λ, δ are dual to k and d, and α such that $\alpha(h) = 2$ and $\alpha(k) = \alpha(d) = 0$. We can choose $\alpha_1 = \alpha$ and $\alpha_2 = \delta - \alpha$. There are other choices, but the resulting realizations for different sets of choices will all be isomorphic. As $\alpha_i(\alpha_1^\vee + \alpha_2^\vee) = a_{i1} + a_{i2} = 0$ for $i = 1, 2$ and k is the only element in the kernel of both chosen α_is, we have that $\alpha_2 = k - h$. If $\mathfrak{h}$ was 2-dimensional it would not be possible to choose linearly independent α_is such that they share a kernel.

From here we need to construct $\mathfrak{n}$ and $\mathfrak{n}^-$. Note that $\mathfrak{n}$ only depends on the one relation $\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0$ for $i \neq j$. The details of constructing $\mathfrak{n}$ through just this relation are technical and we omit them. The root system Δ is illustrated by figure 2.1. Each node is a root. Note, all the roots lie on the α-δ plane.

The dominant chamber is a little different from what we would expect. One of the walls is given by the δ-Λ plane and the other is the plane spanned by δ and $\frac{1}{2}\alpha + \Lambda$. The Tits cone is formed by $\{a\alpha + b\delta + c\Lambda | c > 0\}$.

2.3.2 Representation Theory

The representation theory of Kac-Moody algebras is similar to the finite dimensional case, after some restrictions. We cannot work with all representations, as there are some that do no lift to group representations and some that cannot be classified. We restrict our view to the dominant representations.

Dominant Weights and Verma Modules

The dominant weights of $K(A)$ are:

$$D = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_+ \text{ for all } i \in I\}.$$

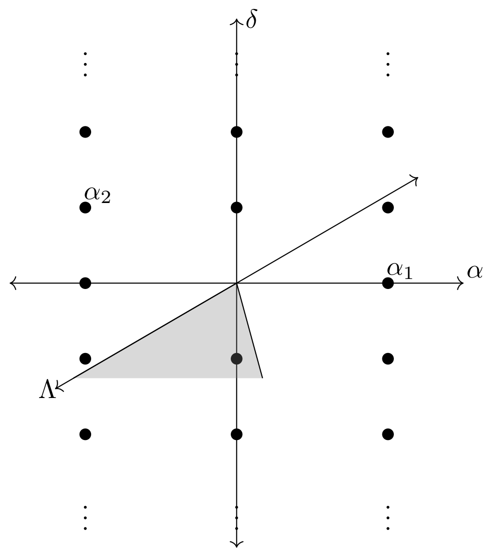

Figure 2.1: Root Diagram associated to $\mathbb{T} \ltimes \tilde{L}SU(2)$. The grey triangle is the dominant chamber intersected with the α-Λ plane.

To each $\lambda \in D$ we may associate an irreducible representation $L(\lambda)$, called an irreducible dominant representation. We will construct $L(\lambda)$ using Verma modules. Recall the universal enveloping algebra for a Lie algebra is $U(\mathfrak{g}) = T(\mathfrak{g})/\sim$ where $g \otimes h - h \otimes g \sim [g, h]$ and $T(\mathfrak{g})$ is the tensor algebra.

Let $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$. By the Poincare-Birkoff-Whit theorem this is isomorphic as a vector space to $U(\mathfrak{n}^{-1}) \otimes \mathbb{C}_\lambda$ and has basis $f_1^{r_1} \dots f_n^{r_n} \otimes \mathbb{1}_\lambda$. M_λ is a $\mathfrak{g}$-module with highest weight λ. It is, however, not irreducible, and is not closed under the Weyl group action. Let M_λ^1 be generated by $f_i^{1+\lambda(\alpha_i^\vee)} \otimes \mathbb{1}_\lambda$. It is the maximal proper submodule of M_λ and thus $L(\lambda) = M_\lambda / M_\lambda^1$ is irreducible. A way to show that these are the correct generators is to consider the following: $f_i^{1+\lambda(\alpha_i^\vee)} \otimes \mathbb{1}_\lambda$ has weight $\lambda - (\lambda(\alpha_i^\vee) + 1)\alpha_i$. We have

$$s_i(\lambda - (\lambda(\alpha_i^\vee) + 1)\alpha_i) = \lambda + \alpha_i,$$

which has higher weight than λ. However, λ is the highest weight, the action of the Weyl group on $L(\lambda)$ would not be closed if $\lambda - (\lambda(\alpha_i^\vee) + 1)\alpha_i \in L(\lambda)$.

2.3.3 The Group Structure $\mathcal{K}(A)$

Given a Kac-Moody algebra $\mathfrak{g}(A)$ we can construct a group $\mathcal{G}(A)$. However, it is not as straight forward as in the finite dimensional case.

A map $T : V \to V$ is locally finite if for all $v \in V$ there is a finite dimensional T-invariant subspace W and $v \in W$. $T : V \to V$ is locally nilpotent if it is also nilpotent. If T is locally finite then it is possible to define $\exp(T) = \sum_{n=0}^{\infty} \frac{T^n}{n!}$.

A representation V of a Kac-Moody algebra is integrable if e_i and f_i act locally nilpotent on V, for all $i \in I$. These are the representations that can be lifted to the group. While there are more representations of the Kac-Moody group than the integrable ones, these are the representations that are easily studied. Also, the space of all integrable representations is exactly the space of dominant representations.

The Standard Parabolic Groups $\mathcal{P}_i$

As $\mathfrak{n}$ is a nilpotent algebra, by Theorem 4.4.19 from [14] we may exponentiate it to obtain a group U. Recall that $\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. Let $J \subset I$ be of finite type. The parabolic algebra $\mathfrak{p}_J$ has two components the nil-radical $\mathfrak{u}_J$ which is locally nilpotent and $\mathfrak{g}_J$ which is finite dimensional. Let $\mathcal{G}_J$ be the complex Lie group associated to $\mathfrak{g}_J$ and let $\exp(\mathfrak{u}_J) = \mathcal{U}_J$. The action of $\mathfrak{g}_J$ on $\mathfrak{u}_J$ lifts to and action of $\mathcal{G}_J$ on $\mathcal{U}_J$. We define the standard parabolic group to be:

$$\mathcal{P}_J = \mathcal{U}_J \ltimes \mathcal{G}_J.$$

Let $\mathcal{B} = \mathcal{P}_\emptyset$, we have $\mathcal{B} \subset \mathcal{P}_J$ for all J. The last two ingredients in building a Tits system, aside from the actual group, are the maximal torus and the normalizer. Let $T = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{C}^*)$. We may define a normalizer N to be an extension of the Weyl group by the torus:

$$0 \to T \to N \to W \to 0,$$

such that $N \cap \mathcal{P}_i = T \cup s_i T$. Then $\mathcal{G}$ is the colimit of the system $(N, P_{\{i\}})_{i \in I}$.

The group $\mathcal{G}$ we have constructed is the complex form of the Kac-Moody group. To form the real form, $\mathcal{K}$ we have to have an involution on $\mathcal{G}$. The algebra $\mathfrak{g}$ carries an anti-linear involution ω where $\omega(h) = \bar{h}$, $\omega(e_i) = -\bar{f}_i$ and $\omega(f_i) = -\bar{e}_i$. This lifts to an involution of $\mathcal{G}$, let $\mathcal{K} = \mathcal{G}^\omega$, the fixed points of $\mathcal{G}$ under ω.

2.4 Dominant K-theory

2.4.1 Representing Space

Let $\mathcal{H}$ be the completed sum of infinitely many copies of each irreducible dominant representation. We call $\mathcal{H}$ the dominant universe.

$$\mathcal{H} = \hat{\bigoplus_{\lambda \in D}} \mathbb{C}^\infty \otimes L(\lambda).$$

Let $\mathcal{F}(\mathcal{H})$ be the Fredholm operators on $\mathcal{H}$, using the same definition as the compact case. For each proper $\mathcal{K}$-space (those with compact isotropy groups) we may define the dominant K-theory to be

$$\mathbb{K}^0_{\mathcal{K}}(X) = [X, \mathcal{F}(\mathcal{H})]_{\mathcal{K}},$$

$$\mathbb{K}^1_{\mathcal{K}}(X) = [X, \Omega\mathcal{F}(\mathcal{H})]_{\mathcal{K}},$$

As $\mathcal{H}$ is maximal, there is a Bott periodicity theorem, so the above is enough to define the theory.

2.4.2 Dominant K-theory of a Compact Subgroup

For each subgroup $H \subset \mathcal{K}$ we may define the dominant K-theory for that subgroup. Let X be a H-space. We may define the dominant K-theory of X with respect to the Cartan matrix A to be:

$${}^A\mathbb{K}^*_H(X) = \mathbb{K}_{\mathcal{K}}(X \wedge_H \mathcal{K}_+).$$

This will depend on H and its inclusion into $\mathcal{K}$. If A is compact the two functors, ${}^A\mathbb{K}_H$ and K_H, are isomorphic. However as the Tits chamber of a Kac-Moody group is a strict subset of the Cartan subalgebra, for some H there are representations of H that do not occur as subrepresentations of $\mathcal{H}$. Therefore if H be a compact Lie group, then $\mathcal{H}$ is a subspace of $\mathcal{H}_H$, a universe for H and unlike the compact case, $\mathcal{H} \neq \mathcal{H}_H$. Let $M(H)$ be a H-invariant summand in $\mathcal{H}_H$ such that $\mathcal{H} \oplus M(H)$ is H stable. Then there is a map

$$St : \mathcal{F}(\mathcal{H}) \oplus \mathcal{F}(M(H)) \to \mathcal{F}(\mathcal{H}_H).$$

Let $\mathbb{M}_H$ be the cohomology theory represented by $\mathcal{F}(M(H))$. Then we have a map $St : {}^A\mathbb{K}^*_H(X) \oplus \mathbb{M}_H(X) \to K^*_H(X)$.

If we take X to be S^0 and ignore $\mathbb{M}_H$ then we get the map ${}^A\mathbb{K}^*_H(S^0) \to R_H[\beta^{\pm}]$ where R_H is the right hand side is the representation theory of H. The left hand side is described by the following definition and lemma.

Definition 2. *${}^{A}DR_H$ is the representation subring of R_H generated by representations which occur as subrepresentaions of $\mathcal{K}$, i.e., $V \in DR_H$ if and only if there exists a $\lambda \in D$ such that $V \subset (L(\lambda))|_H$.*

As we will be working with the parabolic subgroups, instead of writing $DR_{\mathcal{K}_J(A)}$ and ${}^{A}\mathbb{K}^*_{\mathcal{K}_J(A)}$ we will be writing DR_J and ${}^{A}\mathbb{K}^*_J$ respectively.

Lemma 2.4.1. *Given a proper orbit $Y = \mathcal{K}_+ \wedge_H S^0$ for some compact Lie group $H \subset \mathcal{K}$, the is a map*

$$\mathbb{K}^*_{\mathcal{K}}(Y) \oplus \mathbb{M}_H(Y) = {}^{A}\mathbb{K}^*_H(S^0) \oplus \mathbb{M}_H(Y) \to K^*_H(S^0) = R_H[\beta^{\pm}]$$

Furthermore, the image of $\mathbb{K}^*_{\mathcal{K}}(Y)$ is $DR_H[\beta^{\pm}]$.

Proof. See claim 4.6. [11] □

3

Results

Given a proper $\mathcal{K}$-space M, we can ask what it would mean to "complete" $\mathbb{K}^*_{\mathcal{K}}(M)$. Unlike the classical case there is no obvious candidate with respect to which to complete. Initially, we would like to have a topology on $\mathbb{K}^*_{\mathcal{K}}(M)$, say T such that the completion with respect to that topology gives us an isomorphism $\mathbb{K}^*_{\mathcal{K}}(M)^{\wedge}_{T} \cong K^*(M \times_{\mathcal{K}} E\mathcal{K})$. To see why this cannot work we can take M to be the classifying space of proper $\mathcal{K}$-actions. This is also often called $\underline{E}\mathcal{K}$, and, up to homotopy, is defined to be a $\mathcal{K}$-CW complex such that all isotropy groups are compact and for $H \subset \mathcal{K}$ is compact the fixed point space, $\underline{E}\mathcal{K}^H$, is weakly contractible [15]. $\underline{E}\mathcal{K}$ is a terminal object in the category of proper $\mathcal{K}$-spaces. From [11] theorem 2.4, we know that

$$X(A) = \mathrm{hocolim}_{J \in S(A)} \mathcal{K}/\mathcal{K}_J,$$

where $S(A)$ is the poset category under inclusion of subsets $J \subset I$ such that $\mathcal{K}_J(A)$ is of finite type, is the classifying space of proper $\mathcal{K}(A)$ actions. In [11] Nitu Kitchloo calculated that if $\mathcal{K}$ is of compact type, then $\tilde{\mathbb{K}}^*_{\mathcal{K}}(X(A)) \cong R^{\emptyset}_{T}[\beta^{\pm 1}]$ with $R^{\emptyset}_{T}$ concentrated entirely in degree n, where A is of rank $n+1$. In this case, $\tilde{\mathbb{K}}^*_{\mathcal{K}}(X(A))$ is the kernel of an restriction map induced by an inclusion of $\mathcal{K}/T$ into $X(A)$. By degree considerations, the multiplicative structure on $\tilde{\mathbb{K}}^*_{\mathcal{K}}(X(A))$ is trivial. If we took our topology to be induced from an ideal in $R^{\emptyset}_{T}$ then it would be discrete. Therefore $\tilde{\mathbb{K}}^*_{\mathcal{K}}(X(A))$ is already complete. However, $\tilde{\mathbb{K}}^*_{\mathcal{K}}(X(A)) \ncong K^*(X(A) \times_{\mathcal{K}} E\mathcal{K})$ so completion with respect to an ideal will not produce a similar result to the completion theorem.

Given a proper finite $\mathcal{K}$-CW complex X we can use the spectral sequence arising from the skeletal filtration to compute $\mathbb{K}^*_{\mathcal{K}}(X)$ from its cells and gluing maps.

$$E_1^{s,t} = \mathbb{K}^{s+t}_{\mathcal{K}}(X_s, X_{s-1}) \Rightarrow \mathbb{K}^{s+t}_{\mathcal{K}}(X). \tag{3.0.1}$$

As all the cells of a proper $\mathcal{K}$ space have compact isotropy groups are thus of the form $Y = \mathcal{K}_+ \wedge_H S^n$ we know each cell's dominant K-theory, $\mathbb{K}^*_{\mathcal{K}}(Y) = DR_H[\beta^{\pm 1}]$, with DR_H shifted to lie in degree n. These individual cells can be completed with respect to their dimension maps. The purpose of the local completion section is to prove that the completion of DR_H is equal to the completion of the classical representation ring, R_H at it's augmentation ideal. We will find a 1 dimensional representation $\sigma \in DR_H$ such that when inverted we recover the representation ring, i.e. $DR_H[\sigma^{-1}] = R_H$, and use this result to show:

Theorem 3. *There is an isomorphism*

$$(DR_H)^\wedge_{I \cap DR_H} \cong R_H{}^\wedge_I,$$

where I is the augmentation ideal of R_H.

As each (X_s, X_{s-1}) is just a wedge of orbit spaces, $\mathcal{K} \wedge_{H_i} S^s$ we can use Theorem 3 to create a new spectral sequence whose first page is:

$$E_1^{s,*} = \bigoplus_i (R_{H_{i,s}})^\wedge_I[\beta^\pm], \tag{3.0.2}$$

where $(X_s, X_{s-1}) = \bigvee_i (\mathcal{K} \wedge_{H_{i,s}} S^s)$ and the differentials come from the gluing maps as in 3.0.1. However, this spectral sequence a priori is dependent on the choice of $\mathcal{K}$-CW structure we have for X. Different $\mathcal{K}$-CW structures will produce different spectral sequences that may converge to different groups. We would like 3.0.2 to be well defined on proper $\mathcal{K}$-spaces and converge to $K^*(X \times_{\mathcal{K}} E\mathcal{K})$. This leads to the main result of this work.

Theorem 3.2.1. *There is a space $\widehat{\mathcal{F}(\mathcal{H})}$ and a $\mathcal{K}$-equivariant map $c : \mathcal{F}(\mathcal{H}) \to \widehat{\mathcal{F}(\mathcal{H})}$, such that*

$$\mathbb{K}_{\mathcal{K}}(M) \xrightarrow{c} [M, \widehat{\mathcal{F}(\mathcal{H})}]_{\mathcal{K}} \cong K(M \times_{\mathcal{K}} E\mathcal{K})$$

and $\widehat{\mathcal{F}(\mathcal{H})}$ is non-equivariantly homotopic to $\mathcal{F}(\mathcal{H})$. Furthermore the map c lifts to a map of spectra,

$c : \mathbb{KU} \to \widehat{\mathbb{KU}}$ *and fits into the following diagram, where the bottom sequence is short exact:*

$$\begin{tikzcd} & & & \mathbb{K}^*_{\mathcal{K}}(M) \arrow[dl, "c"'] \arrow[d] & \\ 0 \arrow[r] & \varprojlim^1_n \mathbb{K}^*_{\mathcal{K}}(M \times \mathcal{K}^{*n}) \arrow[r] & K^*(M \times_{\mathcal{K}} E\mathcal{K}) \arrow[r] & \varprojlim_n \mathbb{K}^*_{\mathcal{K}}(M \times \mathcal{K}^{*n}) \arrow[r] & 0 \end{tikzcd}$$

As $\mathcal{F}(\mathcal{H})$ is the model for our $\mathcal{K}$-equivariant periodic K-theory spectrum $\mathbb{KU}$, we'll see from the proof that there exists another $\mathcal{K}$-equivariant periodic spectrum, $\widehat{\mathbb{KU}}$ modeled on $\widehat{\mathcal{F}(\mathcal{H})}$ and the map c lifts to these spectra. This gives us that the spectral sequence arising from a skeletal filtration from some $\mathcal{K}$-CW model of X whose first page is 3.0.2 is well defined and converges to $K^*(X \times_{\mathcal{K}} E\mathcal{K})$.

3.1 Completion of DR_H

We will start with a compact parabolic subgroup as it is much easier to work with and as, by the below proposition, all compact subgroups are conjugate to a subgroup of a compact parabolic subgroup. It will be easy to generalize from the parabolic subgroup case to arbitrary compact subgroups.

Proposition 2. *H is conjugate to a subgroup of $\mathcal{K}_J$ for some J of finite type.*

Proof. Consider the proper space $\mathcal{K}/H$, there is a unique up to $\mathcal{K}$-homotopy map

$$f : \mathcal{K}/H \to X(A).$$

The isotropy groups of $X(A)$ are exactly conjugates of $\mathcal{K}_J$ for J of finite type.

$$H = \mathcal{K}_{[e]_H} \subset \mathcal{K}_{f([e]_H)} \subset g^{-1}\mathcal{K}_J g.$$

□

3.1.1 Completion of DR_J for a Compact Parabolic Subgroup

We will start with a diagram to illustrate the proof. See figure 3.1 for the the orbit of the dominant chamber under the action of the Weyl group for $\mathbb{T} \ltimes \tilde{L}SU(2)$. The dark gray chamber is the dominant chamber, the light gray is the Tits cone. Note that the Tits cone does not cover the whole space. Only weights in the top half of the plane will occur in the representations of the Kac-Moody group, so only weights in the top half of the plane will occur in the dominant representation rings of the subgroups. However, the parabolic groups have representations with weights in the lower half of the plane. If we take the parabolic group generated by $J = \{1\}$, we can chose a dominant integral weight such that $\alpha_1(\sigma_J) = 0$ and $\alpha_2(\sigma_J) = 1$. This will project to the marked point σ_J. If we subtract this element from the integral weights that occur in the dominant representations we can obtain all the weights. We can accomplish this in DR_J by inverting the representation associated to σ_1, which happens to be 1-dimensional.

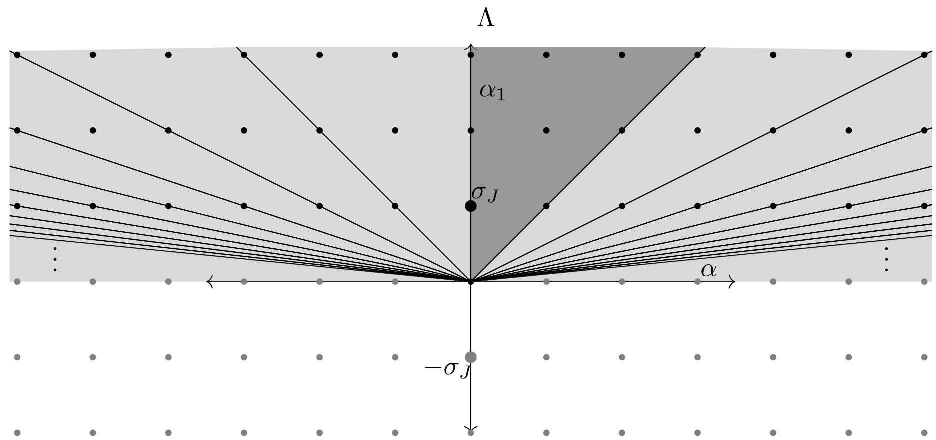

Figure 3.1: The Tits cone and dominant chamber for $\mathbb{T} \ltimes \tilde{L}SU(2)$. Only the Λ-α plane is shown to emphasize that the Tits cone only covers half of the space. The weights that occur in dominant representations are black, the rest are gray.

Lemma 3.1.1. *There exists L a 1-dimensional representation of $\mathcal{K}_J$ such that*

$$DR_J[L^{\pm 1}] = R_{\mathcal{K}_J},$$

where the DR_J is the dominant representation ring associated to $\mathcal{K}_J \subset \mathcal{K}$.

First we prove some intermediate steps. Let D be the dominant chamber for our Kac-Moody algebra. As we are dealing with parabolic groups they have maximal rank and have the same Cartan subalgebra as the Kac-Moody algebra, so they all share the dominant chamber. Let D_J be the following set

$$D_J = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+ \text{ for all } i \in J\}$$

Clearly $D \subset D_J$. Also for the torus, $D_\emptyset$, is the complete integral lattice. The following lemma relates the dominant chamber with the parabolic group's dominant chamber through the Weyl group. Recall that W'_J is a set of shortest length representatives for W/W_J and C is the Tits cone, the orbit of the dominant chamber under the action of W.

Proposition 3. *The set $D_J \cap C$ can be characterized in the following way*

$$\bigcup_{w \in W'_J} w^{-1} D = D_J \cap C.$$

Proof. Let $\lambda \in D_J \cap C$. Then there is a $w \in W$ and $\lambda' \in D$ such that $w^{-1}\lambda' = \lambda$. We need to show that $w \in W'_J$. Suppose $\langle \lambda, \alpha_j^\vee \rangle = 0$ for some $j \in J$. λ is fixed under the action of r_j. Let $\lambda = w^{-1}\lambda'$ as above. Then $\langle \lambda', w r_j \alpha^\vee \rangle = \langle \lambda', w\alpha^\vee \rangle$. Without loss of generality we may assume $l(w) \leq l(wr_j)$. By lemma 1.3.13 in Kummar's book [14] $w\alpha_j \in \Delta^+$.

Suppose $\langle \lambda, \alpha_j^\vee \rangle > 0$ for some $j \in J$. Then for w, λ' as above, we have $0 < \langle \lambda', w\alpha_j^\vee \rangle$. Therefore $w\alpha_j \in \Delta^+$ and by 2.3.1 we have $l(wr_j) \geq l(w)$.

Putting two and two together, if we have $\lambda \in D_J \cap C$ we can choose a $w \in W'_J$ and a $\lambda' \in D$ such that $w^{-1}\lambda' = \lambda$ □

The following argument is a bit clearer than the proof, but only works for the weights on the interior of the dominant chamber, not those on the walls of the dominant chamber. Consider λ such that $\langle \lambda, \alpha_j^\vee \rangle > 0$ for all $j \in J$. Then

$$0 < \langle w^{-1}\lambda', \alpha^\vee \rangle = \langle \lambda', w\alpha^\vee \rangle, \qquad \forall \alpha \in \Delta_J^+.$$

If $w\alpha^\vee \in \Delta^+$ for all $\alpha \in \Delta_J^+$ we are done, $w \in W'_J$

If there is an $\alpha \in \Delta_J^+$ such that $w\alpha \notin \Delta^+$ then $w\alpha \in \Delta^-$ (by triangle decomposition), therefore $\langle \lambda', w\alpha^\vee \rangle \leq 0$, and $\langle \lambda', w\alpha^\vee \rangle = 0$. This is a contradiction, so $w \in W'_J$

The next proposition tells us how we can use W'_J to tell us about the decomposition of a $\mathfrak{g}$ representation into $\mathfrak{g}_J$ representations.

Proposition 4. *Given a $\mathfrak{g}$ representation $L(\lambda)$ with highest weight λ then as a $\mathfrak{g}_J$ representation we have $w^{-1}(\lambda)$ is a highest weight of $L(\lambda)$ for $w \in W'_J$*

Proof. Consider $u \in L(\lambda)_{w^{-1}(\lambda)} \neq 0$ and $n \in \mathfrak{n}_J^+$ with weight $\alpha \in \Delta_J^+$. Suppose $nu \neq 0$, ie is not a highest weight over $\mathfrak{g}_J$. nu has weight $w^{-1}(\lambda) + \alpha$. $w(nu) \neq 0$ with weight $\lambda + w(\alpha)$. As $w(\alpha) \in \Delta^+$, this is a higher weight than λ. This is a contradiction. □

This means that every element of $D_J \cap C$ is a $\mathfrak{g}_J$ highest weight of an $\mathcal{K}_J$-irreducible component of some irreducible representation of $\mathcal{K}$. Therefore DR_J may be identified the representation ring generated by irreducible highest weight representations with highest weight in $D_J \cap C$.

The next proposition gives us a weight such that we can use to recover D_J from $D_J \cap C$. It will also happen to produce a 1 dimensional representation of $\mathcal{K}_J$.

Proposition 5. *There is a $\sigma_J \in D_J$ such that:*

$$D_J = D_J \cap C - \mathbb{Z}\langle \sigma_J \rangle$$

Proof. By 1 we can rewrite the Tits cone as such:

$$C = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* | \langle \lambda, \alpha^\vee \rangle < 0, \text{ only for finitely many } \alpha \in \Delta^{\vee +}\}$$

Consider σ_J such that $\langle \sigma, \alpha_i^\vee \rangle = 1$ for $i \in I - J$, $\langle \sigma_J, \alpha_i^\vee \rangle = 0$ for $i \in J$. σ_J is not unique. For $\alpha \in \Delta^+ - \Delta_J^+$, $\langle -\sigma_J, \alpha^\vee \rangle < 0$, so $-\sigma_J \in D_J - C$. Let $\lambda \in D_J$ and let $c = \max_{i \in I-J}\{-\langle \lambda, \alpha_i^\vee \rangle\}$. Then $\lambda + c\sigma_J \in D$. Therefore

$$D_J = D - \mathbb{Z}\langle \sigma_J \rangle.$$

□

Now that all the pieces are in place we can resume the proof of Lemma 3.1.1. All we need to show is that the σ_J produces a 1 dimensional representation.

Proof of Lemma 3.1.1. The representation associated to σ_J is one dimensional as σ_J is fixed by W_J and the first map in the following exact sequence misses exactly $\pm\sigma_J$

$$\bigoplus_{j \in J} M_J(r_j * (\pm\sigma_J)) \to M_J(\pm\sigma_J) \to L_J(\pm\sigma_J) \to 0.$$

Therefore $L = L_J(\pm\sigma_J)$ lifts to a 1-dimensional representation of $\mathcal{K}_J$ and

$$DR_J[L^{\pm 1}] = R(\mathcal{K}_J).$$

□

3.1.2 Completion of DR_H for any Compact Subgroup

We get the completion of DR_H as a corollary of the following theorem. From it and its corollary we can clearly see the relationship between the dominant representation ring, DR_H, and the representation ring of H. The idea of the proof is to lift the 1 dimensional representation, L_J from $\mathcal{K}_J$ to H

then showing that it is the desired representation.

Theorem 3.1.2. *Let $H \subset \mathcal{K}$ be a compact subgroup. There exists L a 1-dimensional representation of H such that*

$$DR_H[L^{-1}] = R_H,$$

where the DR_H is the dominant representation ring associated to $H \subset \mathcal{K}$. L is well defined up to a unit.

Proof. Without loss of generality, we may assume $H \subset \mathcal{K}_J$ when considering the representation rings. Let $i : H \to \mathcal{K}_J$ be the inclusion. Also, it is obvious that every irreducible representation V of H such that $V \in DR_H$ occurs as a summand of $i^*(W)$ for some $W \in DR_J$. This is because restriction to H from $\mathcal{K}$ factors through i^*.

Let L_J be the same 1-dimensional representation as in the lemma above. Let V be any representation of H, it is easy to show (use Segal's induction) there is an $W \in R_H$ such that $V \subset i^*W$.

$M \otimes L_J^{-m} \in R(\mathcal{K}_J)$ be an irreducible representation where $M \in DR_J$ and $m \in \mathbb{Z}_+$. Restriction respects tensor products so

$$i^*(M \otimes L_J^m) = i^*(M) \otimes i^*(L_J)^{-m}.$$

Let $W \in R_H$ irreducible, such that $W \subset i^*(M) \otimes i^*(L_J)^{-m}$. Then $W \otimes i^*(L_J)^m \subset i^*(M) \in DR_H$, therefore $DR_H[i^*(L_J)^{-1}] \cong R_H$. For brevity in the rest of the proof we will suppress the pullback, i^*.

Now, to show that L is well-defined up to a unit in DR_H. Let $J_1, J_2 \subset I$ such that $H \subset K_{J_1}(A), K_{J_2}(A)$. Then $DR_H[L_{J_1}^{-1}] = R_H = DR_H[L_{J_2}^{-1}]$.

$$\begin{tikzcd} DR_H \arrow[r, "\iota"] \arrow[d] & R_H \\ DR_H[L_{J_2}^{-1}] \arrow[ur] & \end{tikzcd}$$

L_{J_1} gets sent to a unit in R_H, so it must get sent to a unit in $DR_H[L_{J_2}^{-1}]$, all units in $DR_H[L_{J_2}^{-1}]$ are of the form $\eta L_{J_2}^{-1}$, where $\eta \in DR_H$ is a unit. Therefore $L_{J_1} = \eta L_{J_2}$. $\square$

Now that we have this representation we can show that the completion of DR_H and R_H at the augmentation ideal of both results in isomorphic algebras. It's quite simple to show 3 from here.

Proof. For ease of notation let $I_D = I \cap DR_H$. I will show that $DR_H/I_D^n \cong R_H/I^n$.

Consider, in DR_H/I_D^n,

$$(L-1)^n = L^n - nL^{n-1} + \cdots + (-1)^n = 0 \mod I_D^n$$

$$L((-1)^{n+1}(L^{n-1} - nL^{n-2} + \cdots + (-1)^{n-1}n)) = 1 \mod I_D^n.$$

So L maps to a unit under the projection map. We have shown that $R_H = DR_H[L^{\pm 1}]$, so this map factors through R_H by universality.

$$\begin{tikzcd} DR_H \arrow[r] \arrow[d] & R_H \arrow[dl, dotted, "\upsilon_n"] \\ DR_H/I_D^n & \end{tikzcd}$$

Since $\upsilon_n(L^{-1}) = (-1)^{n+1}(L^{n-1} - nL^{n-2} + \cdots + (-1)^{n-1}n)$ has virtual dimension 1. We have for $V \in I$, that $V = \hat{V}L^{-k}$ for some $\hat{V} \in DR_H$ and $k \in \mathbb{N}$. $\hat{V}$ has to have virtual dimension 0, so $\hat{V} \in I_D$. So, we have

$$R_H/I^n = DR_H[L^{-1}]/I^n = (DR_H/I_D^n)[L^{-1}] = DR_H/I_D^n.$$

$\square$

3.2 Completion of $\mathbb{K}_{\mathcal{K}}$

The last step is constructing the object that represents completion. In this section we construct a spectrum $\widehat{\mathbb{KU}}$ to represent geometric completion. This is the spectrum that we use to show that the spectral sequence 3.0.2 is well-defined and converges to the correct object.

3.2.1 Completion of $\mathbb{K}_{\mathcal{K}}^*$

To produce a map of K-theories $c : \mathbb{K}_{\mathcal{K}}^*(M) \to K(M \times_{\mathcal{K}} E\mathcal{K})$ we use a much stronger result. Here, we produce a universal object and map which realize completion. We essentially use the contractability of the unitary group on a separable Hilbert space that was first shown by Nicolaas Kuiper to remove the action of $\mathcal{K}$ on $\mathcal{F}(\mathcal{H})$ [13].

Theorem 3.2.1. *There is a space $\widehat{\mathcal{F}(\mathcal{H})}$ and a $\mathcal{K}$-equivariant map $c : \mathcal{F}(\mathcal{H}) \to \widehat{\mathcal{F}(\mathcal{H})}$, such that*

$$\mathbb{K}_{\mathcal{K}}(M) \xrightarrow{c} [M, \widehat{\mathcal{F}(\mathcal{H})}]_{\mathcal{K}} \cong K(M \times_{\mathcal{K}} E\mathcal{K})$$

and $\widehat{\mathcal{F}(\mathcal{H})}$ is non-equivariantly homotopic to $\mathcal{F}(\mathcal{H})$. The map c extends to a map of K-theories:

$$\mathbb{K}^*_{\mathcal{K}}(M) \xrightarrow{c} K^*(M \times_{\mathcal{K}} E\mathcal{K}).$$

Furthermore the map c fits into the following diagram, where the bottom sequence is short exact:

$$\begin{tikzcd}
 & & & \mathbb{K}^*_{\mathcal{K}}(M) \arrow[dl, "c"'] \arrow[d] & \\
0 \arrow[r] & \varprojlim^1_n \mathbb{K}^*_{\mathcal{K}}(M \times \mathcal{K}^{*n}) \arrow[r] & K^*(M \times_{\mathcal{K}} E\mathcal{K}) \arrow[r] & \varprojlim_n \mathbb{K}^*_{\mathcal{K}}(M \times \mathcal{K}^{*n}) \arrow[r] & 0
\end{tikzcd}$$

Theorem 3.2.2. *There is a space $\widehat{\mathcal{F}(\mathcal{H})}$ and a $\mathcal{K}$-equivariant map $c : \mathcal{F}(\mathcal{H}) \to \widehat{\mathcal{F}(\mathcal{H})}$, such that*

$$\mathbb{K}^{2k}_{\mathcal{K}}(M) \xrightarrow{c} [M, \widehat{\mathcal{F}(\mathcal{H})}]_{\mathcal{K}} \cong K(M \times_{\mathcal{K}} E\mathcal{K}),$$

and $\widehat{\mathcal{F}(\mathcal{H})}$ is non-equivariantly homotopic to $\mathcal{F}(\mathcal{H})$.

Proof. We will produce an equivariant map $\mathcal{F}(\mathcal{H}) \times E\mathcal{K} \xrightarrow{c'} \mathcal{F}(\mathcal{H})^\sharp$, where we regard $\mathcal{F}(\mathcal{H})^\sharp$ to be the same underlying space as $\mathcal{F}(\mathcal{H})$, but with a trivial $\mathcal{K}$-action. As $\mathcal{H}$ is constructed to be a faithful unitary representation of $\mathcal{K}$, the representation map is an embedding $\mathcal{K} \to \mathcal{U}(\mathcal{H})$. We will use $\mathcal{U}$ to refer to $\mathcal{U}(\mathcal{H})$ to simplify the notation. From the representation map we have an induced map $E\mathcal{K} \to E\mathcal{U}$, this allows us to construct the following two maps:

$$\mathcal{F}(\mathcal{H}) \times E\mathcal{K} \xrightarrow{proj} \mathcal{F}(\mathcal{H}) \times_{\mathcal{K}} E\mathcal{K} \to \mathcal{F}(\mathcal{H}) \times_{\mathcal{K}} E\mathcal{U}.$$

Now we may mod out by the larger group $\mathcal{U}$, giving us a map $\mathcal{F}(\mathcal{H}) \times E\mathcal{K} \to \mathcal{F}(\mathcal{H}) \times_{\mathcal{U}} E\mathcal{U}$. The $\mathcal{F}(\mathcal{H}) \times_{\mathcal{U}} E\mathcal{U}$ on the right hand side has a trivial $\mathcal{K}$-action. As $\mathcal{U}$ is contractible $B\mathcal{U}$ is contractible. Thus, the fibration $\mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H}) \times_{\mathcal{U}} E\mathcal{U} \to B\mathcal{U}$ gives us a homotopy equivariance between $\mathcal{F}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H}) \times_{\mathcal{U}} E\mathcal{U}$. However, this is not an $\mathcal{K}$-equivariant homotopy equivilance with the standard action on $\mathcal{F}(\mathcal{H})$. If we regard both sides as having trivial $\mathcal{K}$-action then it is an $\mathcal{K}$-equivariant homotopy equivariance. Thus we have the desired map, $\mathcal{F}(\mathcal{H}) \times E\mathcal{K} \xrightarrow{c'} \mathcal{F}(\mathcal{H})^\sharp$. We may adjoint c' over, $\mathcal{F}(\mathcal{H}) \xrightarrow{c} \mathrm{Hom}(E\mathcal{K}, \mathcal{F}(\mathcal{H})^\sharp)$ sending $f \mapsto g_f$ such that $g_f(e) = c'(f, e)$. Let $\widehat{\mathcal{F}(\mathcal{H})} = \mathrm{Hom}(E\mathcal{K}, \mathcal{F}(\mathcal{H})^\sharp)$. As $E\mathcal{K}$ is non-equivariantly contractible, this is non-equivariantly homotopic to $\mathcal{F}(\mathcal{H})^\sharp$. It is fairly easy to see that $[M, \widehat{\mathcal{F}(\mathcal{H})}]_{\mathcal{K}} \cong K(M \times_{\mathcal{K}} E\mathcal{K})$:

$$[M, \mathrm{Hom}(E\mathcal{K}, \mathcal{F}(\mathcal{H})^\sharp)]_{\mathcal{K}} \cong [M \times E\mathcal{K}, \mathcal{F}(\mathcal{H})^\sharp]_{\mathcal{K}} \cong [M \times_{\mathcal{K}} E\mathcal{K}, \mathcal{F}(\mathcal{H})^\sharp].$$

To extend it to $\mathbb{K}^*_{\mathcal{K}}(M) \xrightarrow{c} K^*(M \times_{\mathcal{K}} E\mathcal{K})$, we just need to produce a map $c : \Omega\mathcal{F}(\mathcal{H}) \to \Omega\widehat{\mathcal{F}(\mathcal{H})}$. This follows from the same argument as above. ☐

Corollary 3.2.2.1. *There is a diagram, where the bottom sequence is short exact.*

$$\begin{tikzcd}
 & & & \mathbb{K}^*_{\mathcal{K}}(M) \arrow[dl, "c"'] \arrow[d] & \\
0 \arrow[r] & \varprojlim^1_n \mathbb{K}^*_{\mathcal{K}}(M \times \mathcal{K}^{*n}) \arrow[r] & K^*(M \times_{\mathcal{K}} E\mathcal{K}) \arrow[r] & \varprojlim_n \mathbb{K}^*_{\mathcal{K}}(M \times \mathcal{K}^{*n}) \arrow[r] & 0
\end{tikzcd} \tag{3.2.1}$$

The short exact sequence on the bottom is the same that occurs in Atiyah and Segal's proof of the completion theorem. However, they show if G is compact then $\varprojlim_n K^*_G(M \times G^{*n})$ satisfies the Mittag-Leffler condition as it is isomorphic to the algebraic completion, $\varprojlim_n K^*_G(M)/K_G(M) \cdot I^n_G$, which is Mittag-Leffer. So, in the compact case $\varprojlim^1 K^*_G(M \times G^{*n})$ is 0. We can not do better than 3.2.1 because there is no candidate for an algebraic completion of $\mathbb{K}_{\mathcal{K}}(M)$ to compare $\varprojlim_n \mathbb{K}_{\mathcal{K}}(M \times \mathcal{K}^{*n})$ against. It still needs to be shown that $\varprojlim^1_n \mathbb{K}_{\mathcal{K}}(M \times \mathcal{K}^{*n})$ is nontrivial in some case.

3.3 Future Directions

To show that the term $\varprojlim^1_n \mathbb{K}_{\mathcal{K}}(M \times \mathcal{K}^{*n})$ in 3.2.1 is non-trivial we need to find an example. In our case the simplest choice to use for M is $X(A)$, the classifying space of proper $\mathcal{K}$-actions. We need to show that the map,

$$K^*(X(A) \times_{\mathcal{K}} E\mathcal{K}) \to \varprojlim_n \mathbb{K}^*_{\mathcal{K}}(X(A) \times \mathcal{K}^{*n}),$$

is not an isomorphism. As $X(A) \times_{\mathcal{K}} E\mathcal{K} \cong B\mathcal{K}$, this will also result in a computation of $K^*(B\mathcal{K})$. The obvious choice for $\mathcal{K}$ is some rank 2 Kac-Moody group, say the affine loop group on $SU(2)$. An alternate description of $X(A)$ is as the topological Tits building,

$$X(A) = \frac{\mathcal{K}/T \times |S(A)|}{\sim},$$

where $(gT, x) \sim (hT, y)$ if $x = y \in \Delta_J$ and $g = h \bmod \mathcal{K}_J$. In the case of a rank 2 Kac-Moody group this gives $X(A)$ a simple $\mathcal{K}$-CW structure with which to compute with. An alternative approach is using the fact that $B\mathcal{K} \cong \mathrm{hocolim}_{|S(A)|} B\mathcal{K}_J$, so we could use the homotopy colimit spectral sequence to compute $K^*(B\mathcal{K})$. The next step would be to compute some of $\varprojlim_n \mathbb{K}_{\mathcal{K}}(X(A) \times \mathcal{K}^{*n})$.

Another direction to go is to attempt to produce another way of computing $K^*(X(A) \times_{\mathcal{K}} E\mathcal{K})$,

Lemma 3.3.1. *For a proper $\mathcal{K}$ space M,*

$$\mathit{hocolim}_{S(A)}(M \times_{\mathcal{K}_J} E\mathcal{K}) = M \times_{\mathcal{K}} EK.$$

Proof.

$$M \times_{\mathcal{K}_J} E\mathcal{K} = \mathrm{hocolim}_S\,(M \times E\mathcal{K}) \times_{\mathcal{K}} (\mathcal{K}/\mathcal{K}_J).$$

Then as taking the product of $\mathcal{K}/\mathcal{K}_J$ with $M \times E\mathcal{K}$ and quotienting by $\mathcal{K}$ are both left adjoints we have:

$$\mathrm{hocolim}_S\,(M \times E\mathcal{K}) \times_{\mathcal{K}} (\mathcal{K}/\mathcal{K}_J) = (M \times E\mathcal{K}) \times_{\mathcal{K}} (\mathrm{hocolim}_S\,\mathcal{K}/\mathcal{K}_J).$$

The last term, $\mathrm{hocolim}_S\,\mathcal{K}/\mathcal{K}_J$, is the topological Tits building and is contractible [6],[10]. So we have

$$(M \times E\mathcal{K}) \times_{\mathcal{K}} (\mathrm{hocolim}_S\,\mathcal{K}/\mathcal{K}_J) \cong (M \times E\mathcal{K}) \times_{\mathcal{K}} (*) \cong M \times_{\mathcal{K}} E\mathcal{K}.$$

□

As shown in section 3.1 we can compute ${}^{A}\mathbb{K}_J(M \times E\mathcal{K})$ for each compact $J \subset I$. We can set up the homotopy colimit spectral sequence from this information. It converges and we have:

$$\lim_{S(A)}^{p}\,{}^{A}\mathbb{K}^*_J(M)^\wedge_{I_J} \Rightarrow K^*(M \times_{\mathcal{K}} E\mathcal{K}).$$

This is an alternate method of computing the completion of Dominant K-theory.

Bibliography

[1] M. Atiyah and G. Segal. Twisted \$K\$-theory. *ArXiv Mathematics e-prints*, July 2004.

[2] M. F. Atiyah and G. B. Segal. Equivariant k-theory and completion. *J. Differential Geom.*, 3(1-2):1–18, 1969.

[3] M.F. Atiyah and D.W. Anderson. *K-theory*. Advanced Books Classics Series. Westview Press, 1994.

[4] Michael F Atiyah. Bott periodicity and the index of elliptic operators. *The Quarterly Journal of Mathematics*, 19.1:pp. 113–140, 1968.

[5] Theodor Bröcker and Tammo tom Dieck. *Representations of Compact Lie Groups*. Springer-Verlag, New York, 1985.

[6] Carles Broto and Nitu Kitchloo. Classifying spaces of kač-moody groups. *Mathematische Zeitschrift*, 240(3):621–649, 2002.

[7] D. Eisenbud. *Commutative Algebra: With a View Toward Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1995.

[8] W. Fulton and J. Harris. *Representation Theory: A First Course*. Graduate Texts in Mathematics / Readings in Mathematics. Springer New York, 1991.

[9] Victor G Kac. Simple irreducible graded lie algebras of finite growth. *Mathematics of the USSR-Izvestiya*, 2(6):1271, 1968.

[10] Nitu Kitchloo. *Topology of Kač-Moody Groups*. PhD thesis, MIT, 5 1998.

[11] Nitu Kitchloo. Dominant k-theory and integrable highest weight representations of kacmoody groups. *Advances in Mathematics*, 221(4):1191 – 1226, 2009.

[12] Nitu Kitchloo. Lectures on loop groups. 2011.

[13] Nicolaas H. Kuiper. The homotopy type of the unitary group of hilbert space. *Topology*, 3(1):19 – 30, 1965.

[14] Shrawan Kumar. *Kac-Moody Groups, Their Flag Varieties and Representation Theory*, volume 204 of *Progress in Mathematics*. Birkhauser, August 2004.

[15] Wolfgang Lück and David Meintrup. On the universal space for group actions with compact isotropy. In *In Geometry and topology: Aarhus*, pages 293–305, 1998.

[16] John Milnor. Construction of universal bundles, ii. *Annals of Mathematics*, 63(3):430–436, 1956.

[17] John Milnor. Construction of universal bundles, ii. *Annals of Mathematics*, 63(3):pp. 430–436, 1956.

[18] R. V. Moody. Lie algebras associated with generalized cartan matrices. *Bull. Amer. Math. Soc.*, 73:217–221, 1967.

[19] A. Pressley and G. Segal. *Loop Groups*. Oxford mathematical monographs. Clarendon Press, 1988.

[20] Graeme Segal. Equivariant k-theory. *Publications Mathématiques de l'IHÉS*, 34:129–151, 1968.

[21] Graeme Segal. The representation ring of a compact lie group. *Publications Mathmatiques de l'Institut des Hautes tudes Scientifiques*, 34(1):113–128, 1968.

Curriculum Vitae

BA with Distinction in Mathematics — Boston University — May 2016

- Awarded inaugural Marvin Freedman prize for excellence in Mathematics

Outreach — Johns Hopkins University — Spring 2016

- Organized the first major outreach for the JHU Mathematics department, a booth at the USA Science and Engineering Festival.
- Had a budget of $2000 to pay for materials, table fee, insurance, travel expenses.
- Scheduled, trained, and lead 10 volunteers over 3 days.

Online instructor for Online Linear Algebra — Johns Hopkins University — Summer 2011, 2012, 2013, 2015

- Introduced Massive Open Online Classroom techniques to significantly improve student outcomes, the average attendance increased to an average of 80% from 12%, failure and dropout rate decreased from 25% to 4%.
- Created over 65 videos and 30 accompanying quizzes to complement the live streamed lectures. The videos were reused in Honors Linear Algebra and the semester Linear Algebra.
- Wrote the exams, syllabi, and administered grades. Trained a co-teacher and managed a grader and a TA for each instance of the class.

Intersession Instructor — Johns Hopkins University — Winter 2014, 2015

- Made online video content to supplement the lack of an appropriate textbook. Researched particular topics and explanations to tailor the content to the students and the timeframe.
- Designed and taught 2 2-credit classes, Lie Algebras and Notions of Proof and Godel's Incompleteness Theorem, for about 30 students total.

Teaching Assistant — Johns Hopkins University — 2010-Present

- Assisted over 12 courses by teaching 23 sections of about 25 students each over 12 semesters. Graded homework and exams, ran office hours, facilitated discussion in section, and lectured on topics the professor missed.

His dissertation was completed under the guidance of Nitu Kitchloo and defended on Thursday, May 26$^{\text{th}}$ 2016.