$$\varDelta = \begin{vmatrix} 1 & , 1 & , \ldots . 1 & , \ldots . 1 \\ \gamma_1 & , \gamma_2 & , \quad \gamma_\rho & , \quad \gamma_p \\ \gamma_1^2 & , \gamma_2^2 & , \quad \gamma_\rho^2 & , \quad \gamma_p^2 \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{p-1}, & \gamma_2^{p-1}, & \ldots . \gamma_\rho^{p-1}, & \ldots . \gamma_p^{p-1} \end{vmatrix} = \prod_{i,j=1}^{i,j=p} (\gamma_i - \gamma_j) \qquad i \neq j$$

is evidently different from zero, the γ's being the p^{th} roots of unity. Of course the value of $\varDelta^2$ is $(-1)^{\frac{(p-1)(p-2)}{2}} p^p$.

Denoting the minor of γ_i^k in $\varDelta$ by $C_{i-1,k}$, the determinant of the minors is

$$D = \begin{vmatrix} C_{0,0} & , C_{1,1} & , \ldots . , & C'_{p-1,0} \\ C_{0,1} & , C_{1,1} & , \ldots . , & C_{p-1,1} \\ \vdots & \vdots & & \vdots \\ C_{0,p-1}, & C_{1,p-1}, & \ldots . , & C_{p-1,p-1} \end{vmatrix}.$$

Solving equations (10), we have

$$\left.\begin{array}{l} \varDelta R_1 = C_{0,0}\zeta_0 + C_{0,1}\zeta_1 + C_{0,2}\zeta_2 + \ldots . + C_{0,p-1}\zeta_{p-1} \\ \varDelta R_2 = C_{1,0}\zeta_0 + C_{1,1}\zeta_1 + C_{1,2}\zeta_2 + \ldots . + C_{1,p-1}\zeta_{p-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \varDelta R_p = C_{p-1,0}\zeta_0 + C_{p-1,1}\zeta_1 + C_{p-1,2}\zeta_2 + \ldots . + C_{p-1,p-1}\zeta_{p-1} \end{array}\right\} \quad (11)$$

The remaining $p^3 - p$ functions $[\lambda, \mu, \nu]$ for $\lambda, \mu = 1, 2, \ldots\ldots, p-1$ and $\nu = 0, 1, 2, \ldots., p-1$ can now be expressed as linear homogeneous functions of

$$\zeta_\nu \qquad \nu = 0, 1, 2, \ldots ., p-1.$$

From (6) it is obvious that

$$\left.\begin{array}{l} R_1(x + \frac{\lambda\omega_1}{p}, y, z) = e^{\lambda \frac{2\pi i}{p}} R_1(x, y, z) = \gamma_\lambda R_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ R_\rho(x + \frac{\lambda\omega_1}{p}, y, z) = e^{\lambda\rho^2 \frac{2\pi i}{p}} R_\rho(x, y, z) = \gamma_\lambda^{\rho^2} R_\rho \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ R_p(x + \frac{\lambda\omega_1}{p}, y, z) = e^{\lambda p^2 \frac{2\pi i}{p}} R_p(x, y, z) = R_p \end{array}\right\} \quad (12)$$

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1

ON A CERTAIN CLASS OF FUNCTIONS ANALOGOUS TO THE THETA FUNCTIONS.

DISSERTATION

PRESENTED TO THE BOARD OF UNIVERSITY STUDIES OF THE
JOHNS HOPKINS UNIVERSITY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

BY

ABRAHAM COHEN

BALTIMORE

1894

PRESS OF
THE FRIEDENWALD COMPANY
BALTIMORE

INTRODUCTION.

M. Appell, in a brief note in the "Annales de la Faculté des Sciences de Marseilles," gives as an example of a function of three variables having a true period and a quasi-period, analogous to the θ-functions, the function

$$\varphi(x, y, z) = \sum_{m=-\infty}^{m=\infty} e^{am^4 + 4xm^3 + 6ym^2 + 4zm},$$

where, in order that the series may be convergent, the real part of a is to be negative.

This function evidently satisfies the conditions

$$\varphi\left(x + \frac{\pi i}{2}, y, z\right) = \varphi\left(x, y + \frac{\pi i}{3}, z\right) = \varphi\left(x, y, z + \frac{\pi i}{2}\right) = \varphi(x, y, z),$$

$$\varphi(x + a, y + 2x + a, z + 3x + 3y + a) = e^{(a + 4x + 6y + 4z)}\varphi(x, y, z),$$

$$6\frac{\partial\varphi}{\partial x} = \frac{\partial^2\varphi}{\partial y\partial z}, \quad 8\frac{\partial\varphi}{\partial y} = 3\frac{\partial^2\varphi}{\partial z^2}.$$

Moreover, as M. Appell shows, these conditions are sufficient to determine $\varphi(x, y, z)$ to within a constant factor.

The object of the present paper is to investigate the properties of this function and of functions derived from it, as well as of others similar to it, pointing out, as far as possible, their analogy to those of the θ-functions. As is not surprising, some of the properties of the latter seem to have no analogues in the case of the functions here considered. In such instances it has been endeavored to assign the reason, as far as possible.

The great difficulty throughout the preparation of this thesis has been the utter poverty of known theorems holding for functions of more than one complex variable. As a consequence, this work has been rather of a tentative nature.

In this, the assistance rendered me by Professor Craig, at whose suggestion this subject was selected, was invaluable, my appreciation of which it is only proper that I express here. I desire also to acknowledge the debt of gratitude I owe to both Professor Craig and Professor Franklin for their interest manifested in my work throughout my entire connection with the Johns Hopkins University.

I.

Let $f(x, y, z)$ be a holomorphic function satisfying the conditions

$$f(x+\omega_1, y, z)=f(x, y+\omega_2, z)=f(x, y, z+\omega_3)=f(x, y, z), \tag{1}$$

$$\begin{aligned} f\Big(x+\frac{2a\omega_1}{\pi i}p,\ y+\frac{3x\omega_2}{\omega_1}p+\frac{3a\omega_2}{\pi i}p^2,\ z+\frac{2y\omega_3}{\omega_2}p+\frac{3x\omega_3}{\omega_1}p^2+\frac{2a\omega_3}{\pi i}p^3\Big) \\ = e^{-ap^4-2\pi i\left(\frac{x}{\omega_1}p^3+\frac{y}{\omega_2}p^2+\frac{z}{\omega_3}p\right)} f(x, y, z), \end{aligned} \tag{2}$$

$$\frac{\partial f}{\partial x}=\frac{\omega_2\omega_3}{2\pi i\omega_1}\frac{\partial^2 f}{\partial y\partial z}, \qquad \frac{\partial f}{\partial y}=\frac{\omega_3^2}{2\pi i\omega_2}\frac{\partial^2 f}{\partial z_2} \tag{3}$$

where ω_1, ω_2, ω_3 are any quantities, real or imaginary,

p any given integer,

a a constant whose real part is negative.

The most general entire function of x, y, z satisfying conditions (1) is given by the Fourier series,

$$f(x, y, z)=\sum_{k=-\infty}^{k=\infty}\sum_{l=-\infty}^{l=\infty}\sum_{n=-\infty}^{n=\infty} C_{k,l,m}\, e^{2\pi i\left(\frac{x}{\omega_1}k+\frac{y}{\omega_2}l+\frac{z}{\omega_3}m\right)} \tag{4}$$

where $C_{k,l,m}$ is independent of x, y, z. In order that $f(x, y, z)$ also satisfy conditions (3), we must have

$$k=lm \quad \text{and} \quad l=m^2$$

or

$$k=m^3 \qquad l=m^2$$

and we now have only the simply infinite series

$$f(x, y, z)=\sum_{m=-\infty}^{m=\infty} C_m\, e^{2\pi i\left(\frac{x}{\omega_1}m^3+\frac{y}{\omega_2}m^2+\frac{z}{\omega_3}m\right)}. \tag{4'}$$

Finally, from (2) we have, on multiplying both sides of the equation by

$$e^{am^4+ap^4+2\pi i\left(\frac{x}{\omega_1}p^3+\frac{y}{\omega_2}p^2+\frac{z}{\omega_3}p\right)}$$

and properly collecting the terms

$$\sum_m C_m e^{a(m+p)^4+2\pi i\left[\frac{x}{\omega_1}(m+p)^3+\frac{y}{\omega_2}(m+p)^2+\frac{z}{\omega_3}(m+p)\right]}=\sum_m C_m e^{am^4+2\pi i\left(\frac{x}{\omega_1}m^3+\frac{y}{\omega_2}m^2+\frac{z}{\omega_3}m\right)}.$$

In order that this equation be satisfied, it is evidently necessary and sufficient that, for all values of m,

$$C_m = C_{m+p}.$$

Hence the most general function of x, y, z satisfying the conditions (1), (2), (3) will be given by

$$f(x, y, z) = \sum_{m=-\infty}^{m=\infty} C_m e^{am^4 + 2\pi i (\frac{x}{\omega_1} m^3 + \frac{y}{\omega_2} m^2 + \frac{z}{\omega_3} m)} \tag{5}$$

with

$$C_m = C_{m+p}.$$

Since by hypothesis the real part of a is negative, this function is holomorphic for all values of x, y, z.

If we write

$$\left.\begin{aligned}
R_1(x, y, z) &= \sum_{k=-\infty}^{k=\infty} e^{a(kp+1)^4 + 2\pi i [\frac{x}{\omega_1}(kp+1)^3 + \frac{y}{\omega_2}(kp+1)^2 + \frac{z}{\omega_3}(kp+1)]} \\
R_2(x, y, z) &= \quad \Sigma e^{a(kp+2)^4 + 2\pi i [\frac{x}{\omega_1}(kp+2)^3 + \frac{y}{\omega_2}(kp+2)^2 + \frac{z}{\omega_3}(kp+2)]} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
R_\rho(x, y, z) &= \quad \Sigma e^{a(kp+\rho)^4 + 2\pi i [\frac{x}{\omega_1}(kp+ \;,^3 + \frac{y}{\omega_2}(kp+\rho)^2 + \frac{z}{\omega_3}(kp+\rho)]} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
R_p(x, y, z) &= \quad \Sigma e^{a(kp)^4 + 2\pi i [\frac{x}{\omega_1}(kp)^3 + \frac{y}{\omega_2}(kp)^2 + \frac{z}{\omega_3}(kp)]}
\end{aligned}\right\} \tag{6}$$

it is clear that $f(x, y, z)$ will be a linear homogeneous function of $R_1, R_2, \ldots,$ $R_\rho, \ldots R_p$. Moreover, the latter are linearly independent, as may be seen at once from their development. They can, however, be replaced by simpler functions. Write

$$\varphi(x, y, z) = \sum_{m=-\infty}^{m=\infty} e^{am^4 + 2\pi i (\frac{x}{\omega_1} m^3 + \frac{y}{\omega_2} m^2 + \frac{z}{\omega_3} m)}. \tag{7}$$

Then, for λ, μ, ν any integers, we have

$$\varphi\left(x + \frac{\lambda\omega_1}{p}, y + \frac{\mu\omega_2}{p}, z + \frac{\nu\omega_3}{p}\right) = \sum_m e^{am^4 + 2\pi i (\frac{x}{\omega_1} m^3 + \frac{y}{\omega_2} m^2 + \frac{z}{\omega_3} m) + \frac{2\pi i}{p}(\lambda m^3 + \mu m^2 + \nu m)}$$

or

$$\begin{aligned}
\varphi\left(x + \frac{\lambda\omega_1}{p}, y + \frac{\mu\omega_2}{p}, z + \frac{\nu\omega_3}{p}\right) = e^{\frac{2\pi i}{p}(\lambda + \mu + \nu)} R_1 + e^{\frac{2\pi i}{p}(8\lambda + 4\mu + 2\nu)} R_2 + \cdots \\
+ e^{\frac{2\pi i}{p}(\lambda\rho^3 + \mu\rho^2 + \nu\rho)} R_\rho + \cdots + R_p.
\end{aligned} \tag{8}$$

Giving to λ, μ, ν each separately all values from 0 to $p-1$ we get p^3 equations of the type (8) to be satisfied by the p quantities R. Of these p^3 equations, only p can be independent. Moreover there *are* p independent ones among them, viz. as we shall see, those obtained by putting $\lambda=\mu=0$ and letting ν take all integer values from 0 to $p-1$; these are

$$\varphi\left(x,\ y,\ z+\frac{\nu\omega_3}{p}\right)=e^{\frac{2\pi i\nu}{p}}R_1+\ \ldots\ +e^{\frac{2\pi i\rho\nu}{p}}R_\rho+\ \ldots\ +R_p \qquad \nu=0,\ 1,\ \ldots,\ p-1. \tag{9}$$

For the sake of brevity we shall introduce the following notation. Write

$$\begin{aligned}
\varphi(x,\ y,\ z)&=[0,\ 0,\ 0]\\
\varphi\left(x+\frac{\lambda\omega_1}{p},\ y,\ z\right)&=[\lambda,\ 0,\ 0]\\
\varphi\left(x,\ y+\frac{\mu\omega_2}{p},\ z\right)&=[0,\ \mu,\ 0]\\
\varphi\left(x,\ y,\ z+\frac{\nu\omega_3}{p}\right)&=[0,\ 0,\ \nu]\\
\varphi\left(x+\frac{\lambda\omega_1}{p},\ y+\frac{\mu\omega_2}{p},\ z\right)&=[\lambda,\ \mu,\ 0]\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
\varphi\left(x+\frac{\lambda\omega_1}{p},\ y+\frac{\mu\omega_2}{p},\ z+\frac{\nu\omega_3}{p}\right)&=[\lambda,\ \mu,\ \nu].
\end{aligned}$$

Also write

$$[0,\ 0,\ \nu]=\zeta_\nu(x,\ y,\ z)=\zeta_\nu.$$

Our equations (9) may then be written

$$\left.\begin{aligned}
[0,\ 0,\ 0]&=\zeta_0=R_1+R_2+\ \ldots\ +R_\rho+\ \ldots\ +R_p\\
[0,\ 0,\ 1]&=\zeta_1=\gamma_1R_1+\gamma_2R_2+\ \ldots\ +\gamma_\rho R_\rho+\ \ldots\ +\gamma_pR_p\\
[0,\ 0,\ 2]&=\zeta_2=\gamma_1^2R_1+\gamma_2^2R_2+\ \ldots\ +\gamma_\rho^2R_\rho+\ \ldots\ +\gamma_p^2R_p\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
[0,\ 0,\ p-1]&=\zeta_{p-1}=\gamma_1^{p-1}R_1+\gamma_2^{p-1}R_2+\ldots+\gamma_\rho^{p-1}R_\rho+\ldots+\gamma_p^{p-1}R_p
\end{aligned}\right\} \tag{10}$$

where

$$\gamma_1=e^{\frac{2\pi i}{p}},\ \gamma_2=e^{\frac{4\pi i}{p}},\ \ldots,\ \gamma_\rho=e^{\frac{2\rho\pi i}{p}},\ \ldots,\ \gamma_p=1.$$

The independence of these equations follows at once from the fact that the determinant of the system

$$\varDelta=\begin{vmatrix} 1 & , 1 & , \ldots . 1 & , \ldots . 1 \\ \gamma_1 & , \gamma_2 & , \quad \gamma_\rho & , \quad \gamma_p \\ \gamma_1^2 & , \gamma_2^2 & , \quad \gamma_\rho^2 & , \quad \gamma_p^2 \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{p-1}, & \gamma_2^{p-1}, & \ldots . \gamma_\rho^{p-1}, & \ldots . \gamma_p^{p-1} \end{vmatrix} = \prod_{i,j=1}^{i,j=p} (\gamma_i-\gamma_j) \qquad i \neq j$$

is evidently different from zero, the γ's being the p^{th} roots of unity. Of course the value of $\varDelta^2$ is $(-1)^{\frac{(p-1)(p-2)}{2}} p^p$.

Denoting the minor of γ_i^k in $\varDelta$ by $C_{i-1,k}$, the determinant of the minors is

$$D=\begin{vmatrix} C_{0,0} & , C_{1,1} & , \ldots . , & C_{p-1,0} \\ C_{0,1} & , C_{1,1} & , \ldots . , & C_{p-1,1} \\ \vdots & \vdots & & \vdots \\ C_{0,p-1}, & C_{1,p-1}, & \ldots . , & C_{p-1,p-1} \end{vmatrix}.$$

Solving equations (10), we have

$$\left.\begin{array}{l} \varDelta R_1 = C_{0,0}\zeta_0 + C_{0,1}\zeta_1 + C_{0,2}\zeta_2 + \ldots . + C_{0,p-1}\zeta_{p-1} \\ \varDelta R_2 = C_{1,0}\zeta_0 + C_{1,1}\zeta_1 + C_{1,2}\zeta_2 + \ldots . + C_{1,p-1}\zeta_{p-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \varDelta R_p = C_{p-1,0}\zeta_0 + C_{p-1,1}\zeta_1 + C_{p-1,2}\zeta_2 + \ldots . + C_{p-1,p-1}\zeta_{p-1} \end{array}\right\} \quad (11)$$

The remaining p^3-p functions $[\lambda, \mu, \nu]$ for $\lambda, \mu=1, 2, \ldots\ldots, p-1$ and $\nu=0, 1, 2, \ldots., p-1$ can now be expressed as linear homogeneous functions of

$$\zeta_\nu \qquad \nu=0, 1, 2, \ldots ., p-1.$$

From (6) it is obvious that

$$\left.\begin{array}{l} R_1(x+\frac{\lambda\omega_1}{p}, y, z) = e^{\lambda\frac{2\pi i}{p}} R_1(x, y, z) = \gamma_\lambda R_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ R_\rho(x+\frac{\lambda\omega_1}{p}, y, z) = e^{\lambda\rho^3\frac{2\pi i}{p}} R_\rho(x, y, z) = \gamma_\lambda^{\rho^3} R_\rho \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ R_p(x+\frac{\lambda\omega_1}{p}, y, z) = e^{\lambda p^3\frac{2\pi i}{p}} R_p(x, y, z) = R_p \end{array}\right\} \quad (12)$$

Making these changes in (11) we get the following p systems of $p-1$ equations each:

$$\gamma_\lambda \varDelta R_1 = C_{0,0}[\lambda,\ 0,\ 0] + C_{0,1}[\lambda,\ 0,\ 1] + \dots + C_{0,p-1}[\lambda,\ 0,\ p-1] \quad (13_1)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\gamma_\lambda^{\rho^3} \varDelta R_\rho = C_{\rho-1,0}[\lambda,\ 0,\ 0] + C_{\rho-1,1}[\lambda,\ 0,\ 1] + \dots + C_{\rho-1,p-1}[\lambda,\ 0,\ p-1] \quad (13_\rho)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\varDelta R_p = C_{p-1,0}[\lambda,\ 0,\ 0] + C_{p-1,1}[\lambda,\ 0,\ 1] + \dots + C_{p-1,p-1}[\lambda,\ 0,\ p-1] \quad (13_p)$$

$$\lambda = 1,\ 2,\ \dots,\ p-1.$$

The determinant of each of the systems of p equations obtained by taking the λ^{th} equation of each of the above sets is D, which is different from zero since $\varDelta$ is.

Hence we can solve for the p^2-p quantities $[\lambda,\ 0,\ \nu]$ for $\lambda = 1,\ \dots,\ p-1$ and $\nu = 0,\ 1,\ \dots,\ p-1$ in terms of $R_1,\ R_2,\ \dots,\ R_p$, which, in turn, can be expressed linearly in terms of

$$\zeta_\nu \qquad \nu = 0,\ 1,\ \dots\ p-1.$$

Thus, taking the system

$$\left.\begin{aligned}
\gamma_\lambda \varDelta R_1 &= C_{0,0}[\lambda,\ 0,\ 0] + C_{0,1}[\lambda,\ 0,\ 1] + \dots + C_{0,p-1}[\lambda,\ 0,\ p-1]\\
\gamma_\lambda^{2^3} \varDelta R_2 &= C_{1,0}[\lambda,\ 0,\ 0] + C_{1,1}[\lambda,\ 0,\ 1] + \dots + C_{1,p-1}[\lambda,\ 0,\ p-1]\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
\gamma_\lambda^{\rho^3} \varDelta R_\rho &= C_{\rho-1,0}[\lambda,\ 0,\ 0] + C_{\rho-1,1}[\lambda,\ 0,\ 1] + \dots + C_{\rho-1,p-1}[\lambda,\ 0,\ p-1]\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
\varDelta R_p &= C_{p-1,0}[\lambda,\ 0,\ 0],\quad C_{p-1,1}[\lambda,\ 0,\ 1] + \dots + C_{p-1,p-1}[\lambda,\ 0,\ p-1]
\end{aligned}\right\}$$

and remembering that the minor of $C_{i,k}$ in D is $\gamma_{i+1}^k \varDelta^{p-2}$, we have

$$[\lambda,\ 0,\ \nu] = \sum_{\rho=1}^{\rho=p} \gamma_\lambda^{\rho^3} \gamma_\rho^\nu R_\rho.$$

And finally, from (11)

$$\varDelta[\lambda,\ 0,\ \nu] = \sum_{\rho=1}^{\rho=p} \sum_{j=0}^{j=p-1} \gamma_\lambda^{\rho^3} \gamma_\rho^\nu C_{\rho-1,j} \zeta_j \tag{14}$$

$$\lambda = 1,\ 2,\ \dots,\ p-1 \qquad \nu = 0,\ 1,\ 2,\ \dots,\ p-1.$$

In exactly the same way we get

$$\varDelta[0,\ \mu,\ \nu] = \sum_{\rho=1}^{\rho=p} \sum_{j=0}^{j=p-1} \gamma_\mu^{\rho^2} \gamma_\rho^\nu C_{\rho-1,j} \zeta_j \tag{15}$$

$$\mu = 1,\ 2,\ \dots,\ p-1 \qquad \nu = 0,\ 1,\ 2,\ \dots,\ p-1$$

or

$$\varDelta[0,\ \mu,\ \nu] = \sum_{\rho=1}^{\rho=p} \gamma_\mu^{\rho^2}\gamma_\rho^\nu C_{\rho-1,\,0}[0,\ 0,\ 0] + \sum_{\rho=1}^{\rho=p} \gamma_\mu^{\rho^2}\gamma_\rho^\nu C_{\rho-1,\,1}[0,\ 0,\ 1] + \cdots$$

$$+ \cdots + \Sigma\gamma_\mu^{\rho^2}\gamma_\rho^\nu C_{\rho-1,\,\sigma}[0,\ 0,\ \sigma] + \cdots + \Sigma\gamma_\mu^{\rho^2}\gamma_\rho^\nu C_{\rho-1,\,p-1}[0,\ 0,\ p-1].$$

$$\therefore\quad \varDelta[\lambda,\ \mu,\ \nu] = \sum_{\rho=1}^{\rho=p} \gamma_\mu^{\rho^2}\gamma_\rho^\nu C_{\rho-1,\,0}[\lambda,\ 0,\ 0] + \Sigma\gamma_\mu^{\rho^2}\gamma_\rho^\nu C_{\rho-1,\,1}[\lambda,\ 0,\ 1] + \cdots$$

$$+ \cdots + \Sigma\gamma_\mu^{\rho^2}\gamma_\rho^\nu C_{\rho-1,\,\sigma}[\lambda,\ 0,\ \sigma] + \cdots + \Sigma\gamma_\mu^{\rho^2}\gamma_\rho^\nu C_{\rho-1,\,p-1}[\lambda,\ 0,\ p-1].$$

Whence, from (14) we have

$$\begin{aligned}\varDelta^2[\lambda,\ \mu,\ \nu] = &\sum_{\rho=1}^{\rho=p} \gamma_\mu^{\rho^2}\gamma_\rho^\nu C_{\rho-1,\,0} \sum_{\rho'=1}^{\rho'=p} \sum_{j=0}^{j=p-1} \gamma_\lambda^{\rho'^3} C_{\rho'-1,\,j}\,\zeta_j \\ &+ \Sigma\,\gamma_\mu^{\rho^2}\gamma_\rho^\nu C_{\rho-1,\,1}\Sigma\,\Sigma\,\gamma_\lambda^{\rho'^3}\gamma_{\rho'} C_{\rho'-1,\,j}\,\zeta_j \\ &+ \Sigma\,\gamma_\mu^{\rho^2}\gamma_\rho^\nu C_{\rho-1,\,2}\Sigma\,\Sigma\,\gamma_\lambda^{\rho'^3}\gamma_{\rho'}^2 C_{\rho'-1,\,j}\,\zeta_j \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots \\ &+ \Sigma\,\gamma_\mu^{\rho^2}\gamma_\rho^\nu C_{\rho-1,\,p-1}\Sigma\,\Sigma\,\gamma_\lambda^{\rho'^3}\gamma_{\rho'}^{p-1} C_{\rho'-1,\,j}\,\zeta_j\end{aligned} \tag{16}$$

$$\lambda,\ \mu = 1,\ 2,\ \ldots,\ p-1 \qquad \nu = 0,\ 1,\ 2,\ \ldots,\ p-1.$$

If we write (14) in the form

$$[\lambda,\ 0,\ \nu] = \frac{1}{\varDelta}\sum_{\rho=1}^{\rho=p}\sum_{j=0}^{j=p-1} \gamma_\lambda^{\rho^3}\gamma_\rho^\nu C_{\rho-1,\,j}\,\zeta_j \equiv \frac{1}{\varDelta}\sum_{j=0}^{j=p-1} A_{j,\,\nu}^{(\lambda)}\,\zeta_j \tag{17}$$

the determinant of any of the systems of p equations obtained by keeping λ fixed and allowing ν to take all values from 0 to $p-1$ is seen at once on writing it out, to be

$$\gamma_\lambda^{(1^3+2^3+\cdots+p^3)}\frac{D\varDelta}{\varDelta^p}$$

or since $D = \varDelta^{p-1}$,

$$\gamma^{[\frac{1}{2}p\,(p+1)]^2} \equiv \frac{1}{\varDelta^p}\left| A_{j,\,\nu}^{(\lambda)} \right|.$$

Similarly, writing (15) in the form

$$[0,\ \mu,\ \nu] = \frac{1}{\varDelta}\sum_{\rho=1}^{\rho=p}\sum_{j=0}^{j=p-1} \gamma_\mu^{\rho^2}\gamma_\rho^\nu C_{\rho-1,\,j}\,\zeta_j \equiv \frac{1}{\varDelta}\sum_{j=0}^{j=p-1} B_{j,\,\nu}^{(\mu)}\,\zeta_j \tag{18}$$

the determinant of any of the systems of p equations obtained by keeping μ fixed is found to be

$$\gamma_\mu^{(1^2+2^2+\cdots+p^2)} = \gamma_\mu^{\frac{p\,(p-1)\,(2p-1)}{6}} \equiv \frac{1}{\varDelta^p}\left| B_{j,\,\nu}^{(\mu)} \right|.$$

Finally, (16) may be put in the form

$$[\lambda,\ \mu,\ \nu] = \frac{1}{\Delta^2} \sum_{k=0}^{k=p-1} \sum_{j=0}^{j=p-1} A^{(\lambda)}_{k,\ \nu} B^{(\mu)}_{j,\ k} \zeta_j. \tag{19}$$

The coefficients here are, to within the factor $\frac{1}{\Delta^2}$, the elements of the determinant obtained by taking the product

$$|A^{(\lambda)}_{k,\ \nu}| \cdot |B^{(\mu)}_{j,\ k}| = \Delta^{2p} \gamma_1^{\lambda \left[\frac{p(p+1)}{2}\right]^2 + \mu \frac{p(p-1)(2p-1)}{6}}.$$

We have thus expressed all of the p^3 quantities

$$[\lambda,\ \mu,\ \nu] \qquad \lambda,\ \mu,\ \nu = 0,\ 1,\ 2, \ldots p-1$$

as linear homogeneous functions of the p linearly independent ones

$$[0,\ 0,\ \nu] \equiv \zeta_\nu \qquad \nu = 0,\ 1,\ \ldots p-1.$$

Hence we see that every holomorphic function of x, y, z satisfying conditions (1), (2), (3) can be expressed as a linear homogeneous function of these p quantities. From which follows that there can be only p such functions which shall be linearly independent.

We have obviously

$$\left.\begin{aligned} \zeta_j(x,\ y,\ z) &= \varphi(x,\ y,\ z + \frac{j\omega_3}{p}) \\ \zeta_j(x,\ y,\ z+\omega_3) &= \zeta_j(x,\ y,\ z) \\ \zeta_j(x,\ y,\ z + k\frac{\omega_3}{p}) &= \zeta_{j+k}(x,\ y,\ z) \end{aligned}\right\}. \tag{20}$$

If for brevity we write

$$ap^4 + 2\pi i\left[\frac{x}{\omega_1}p^3 + \frac{y}{\omega_2}p^2 + \frac{z}{\omega_3}p\right] = E(p)$$

and if we denote the substitution

$$(x,\ y,\ z;\ x + \frac{2a\omega_1}{\pi i}p,\ y + \frac{3x\omega_2}{\omega_1}p + \frac{3a\omega_2}{\pi i}p^2,\ z + \frac{2y\omega_3}{\omega_2}p + \frac{3x\omega_3}{\omega_1}p^2 + \frac{2a\omega_3}{\pi i}p^3)$$

by S_p, we also have

$$\left.\begin{aligned} S_p \zeta_j(x,\ y,\ z) &= e^{-E(p)} \zeta_j(x,\ y,\ z) \\ S_\rho \zeta_j(x,\ y,\ z) &= e^{-E(1) - \rho j \frac{2\pi i}{p}} \zeta_j(x,\ y,\ z) \end{aligned}\right\}. \tag{21}$$

In general, if

$$\chi_{\lambda,\,\mu,\,\nu}(x,\ y,\ z) = \varphi\left(x + \frac{\lambda\omega_1}{p},\ y + \frac{\mu\omega_2}{p},\ z + \frac{\nu\omega_3}{p}\right), \text{ then}$$

$$\chi_{\lambda,\,\mu,\,\nu}(x+\omega_1,\ y,\ z) = \chi_{\lambda,\,\mu,\,\nu}(x,\ y+\omega_2,\ z) = \chi_{\lambda,\,\mu,\,\nu}(x,\ y,\ z+\omega_3) = \chi_{\lambda,\,\mu,\,\nu}(x,\ y,\ z)$$

$$\chi_{\lambda,\,\mu,\,\nu}\left(x + \frac{\lambda'\omega_1}{p},\ y + \frac{\mu'\omega_2}{p},\ z + \frac{\nu'\omega_3}{p}\right) = \chi_{\lambda+\lambda',\,\mu+\mu',\,\nu+\nu'}(x,\ y,\ z)$$

$$S_p\,\chi_{\lambda,\,\mu,\,\nu}(x,\ y,\ z) = e^{-E(p)}\,\chi_{\lambda,\,\mu,\,\nu}(x,\ y,\ z)$$

while the effect of the substitution S_ρ, where $\rho \not\equiv 0 \pmod{p}$, is to change $\chi_{\lambda,\,\mu,\,\nu}(x,\ y,\ z)$ into some other function altogether, in general.

II.

Let us consider now, in connection with the function

$$\zeta_0 = \varphi(x,\ y,\ z) = \sum_{m=-\infty}^{m=\infty} e^{am^4 + 2\pi i\left(\frac{x}{\omega_1} m^3 + \frac{y}{\omega_2} m^2 + \frac{z}{\omega_3} m\right)} \tag{1}$$

the functions obtained by increasing z by $\frac{\omega_3}{4}$, $\frac{\omega_3}{2}$ and by $\frac{3\omega_3}{4}$ respectively.

We may write these functions briefly

$$\left.\begin{aligned}
\varphi(x,\ y,\ z) &= \varphi_0(x,\ y,\ z) = \sum_{m=-\infty}^{m=\infty} e^{E(m)} \\
\varphi\left(x,\ y,\ z + \frac{\omega_3}{4}\right) &= \varphi_1(x,\ y,\ z) = \sum (i)^m\, e^{E(m)} \\
\varphi\left(x,\ y,\ z + \frac{\omega_3}{2}\right) &= \varphi_2(x,\ y,\ z) = \sum (-1)^m\, e^{E(m)} \\
\varphi\left(x,\ y,\ z + \frac{3\omega_3}{4}\right) &= \varphi_3(x,\ y,\ z) = \sum (-i)^m\, e^{E(m)}
\end{aligned}\right\} . \tag{2}$$

From what has preceded, it is plain that the functions obtained by adding to x and to y, respectively, in (1) any multiples of the quarter periods corresponding to them, will be linear homogeneous functions of the four functions (2). In particular, it is obvious that

$$\left.\begin{aligned}
&\varphi\left(x + \frac{\omega_1}{2},\ y,\ z\right) = \varphi\left(x,\ y + \frac{\omega_2}{2},\ z\right) = \varphi\left(x,\ y,\ z + \frac{\omega_3}{2}\right) = \varphi_2(x,\ y,\ z) \\
&\text{and} \\
&\varphi\left(x,\ y + \frac{\omega_2}{2},\ z + \frac{\omega_3}{2}\right) = \varphi\left(x + \frac{\omega_1}{2},\ y,\ z + \frac{\omega_3}{2}\right) = \varphi\left(x + \frac{\omega_1}{2},\ y + \frac{\omega_2}{2},\ z\right) = \varphi_0(x,\ y,\ z)
\end{aligned}\right\} \tag{3}$$

Further, we have manifestly

$$\left.\begin{aligned}
\varphi_j(x+\omega_1,\ y,\ z) = \varphi_j(x,\ y+\omega_2,\ z) = \varphi_j(x,\ y,\ z+\omega_3) &= \varphi_j(x,\ y,\ z) \\
S_1\,\varphi_j(x,\ y,\ z) &= (-i)^j\, e^{-E(1)}\,\varphi_j(x,\ y,\ z) \\
S_p\,\varphi_j(x,\ y,\ z) &= (-i)^{jp}\, e^{E(p)}\,\varphi_j(x,\ y,\ z) \\
j = 0,\ 1,\ 2,\ 3. &
\end{aligned}\right\} \tag{4}$$

If we apply the substitution

$$S_{\frac{1}{2}} = \left(x,\ y,\ z;\ x + \frac{a\omega_1}{\pi i},\ y + \frac{3x\omega_2}{2\omega_1} + \frac{3a\omega_2}{4\pi i},\ z + \frac{y\omega_3}{\omega_2} + \frac{3x\omega_3}{4\omega_1} + \frac{a\omega_3}{4\pi i}\right)$$

we shall get entirely new functions, for

$$\left.\begin{aligned}
S_{\frac{1}{2}}\varphi_0(x,\ y,\ z) &= e^{-E(\frac{1}{2})} \sum_{m=-\infty}^{m=\infty} e^{a(m+\frac{1}{2})^4 + 2\pi i\left[\frac{x}{\omega_1}(m+\frac{1}{2})^3 + \frac{y}{\omega_2}(m+\frac{1}{2})^2 + \frac{z}{\omega_3}(m+\frac{1}{2})\right]} \\
S_{\frac{1}{2}}\varphi_1(x,\ y,\ z) &= e^{-E(\frac{1}{2})} \sum (i)^m e^{a(m+\frac{1}{2})^4 + 2\pi i\left[\frac{x}{\omega_1}(m+\frac{1}{2})^3 + \frac{y}{\omega_2}(m+\frac{1}{2})^2 + \frac{z}{\omega_3}(m+\frac{1}{2})\right]} \\
S_{\frac{1}{2}}\varphi_2(x,\ y,\ z) &= e^{-E(\frac{1}{2})} \sum (-1)^m e^{a(m+\frac{1}{2})^4 + 2\pi i\left[\frac{x}{\omega_1}(m+\frac{1}{2})^3 + \frac{y}{\omega_2}(m+\frac{1}{2})^2 + \frac{z}{\omega_3}(m+\frac{1}{2})\right]} \\
S_{\frac{1}{2}}\varphi_3(x,\ y,\ z) &= e^{-E(\frac{1}{2})} \sum (-i)^m e^{a(m+\frac{1}{2})^4 + 2\pi i\left[\frac{x}{\omega_1}(m+\frac{1}{2})^3 + \frac{y}{\omega_2}(m+\frac{1}{2})^2 + \frac{z}{\omega_3}(m+\frac{1}{2})\right]}
\end{aligned}\right\} \tag{5}$$

This suggests the following functions, which may be written briefly

$$\left.\begin{aligned}
\psi_0(x,\ y,\ z) &= \sum_{m=-\infty}^{m=\infty} e^{E(m+\frac{1}{2})} \\
\psi_1(x,\ y,\ z) &= \sum (i)^m e^{E(m+\frac{1}{2})} \\
\psi_2(x,\ y,\ z) &= \sum (-1)^m e^{E(m+\frac{1}{2})} \\
\psi_3(x,\ y,\ z) &= \sum (-i)^m e^{E(m+\frac{1}{2})}
\end{aligned}\right\} \tag{6}$$

Writing

$$e^{\frac{7}{4}\pi i} = \lambda \qquad e^{\frac{1}{4}\pi i} = \mu$$

we find at once

$$\left.\begin{aligned}
\psi_j(x+\omega_1,\ y+\omega_2,\ z+\omega_3) &= \lambda\psi_{j+1}(x,\ y,\ z) \\
\psi_j(x+\omega_1,\ y,\ z) &= \mu\psi_{j+1}(x,\ y,\ z) \\
\psi_j(x,\ y+\omega_2,\ z) &= i\psi_j(x,\ y,\ z) \\
\psi_j(x,\ y,\ z+\omega_3) &= -\psi_j(x,\ y,\ z) \\
\psi_j(x,\ y,\ z+\tfrac{\omega_3}{4}) &= \mu\psi_{j+1}(x,\ y,\ z) \\
j = 0,\ 1,\ 2,\ 3 \text{ and } \psi_4 &\equiv \psi_0
\end{aligned}\right\} \tag{7}$$

while $\psi_j(x+\frac{\omega_1}{2},\ y,\ z)$ and $\psi_j(x+\frac{\omega_1}{4},\ y,\ z)$ are entirely new functions not expressible as linear combinations of any of the functions φ_j or ψ_j.

It will be seen from these equations, that the periods of ψ_j are not the same as those of φ_j; for, in the case of the former,

$$\left.\begin{array}{llllll} 8\omega_1 & \text{is} & \text{the} & \text{period} & \text{corresponding to} & x \\ 4\omega_2 & `` & `` & `` & `` \quad `` & y \\ 2\omega_3 & `` & `` & `` & `` \quad `` & z \end{array}\right\} \tag{8}$$

although each of the substitutions

$$(x, y, z;\ x+4\omega_1, y, z),\ (x, y, z;\ x, y+2\omega_2, z),\ (x, y, z;\ x, y, z+\omega_3) \tag{8'}$$

operating on ψ_j has only the effect of changing its sign. The effect of the substitution S_p on $\psi_j(x, y, z)$ is the same as that on $\varphi_j(x, y, z)$, viz:

$$\left.\begin{aligned} S_1\psi_j(x, y, z) &= (-i)^j e^{-E(1)}\psi_j(x, y, z) \\ S_p\psi_j(x, y, z) &= (-i)^{jp} e^{-E(p)}\psi_j(x, y, z) \end{aligned}\right\} \tag{9}$$

But while

$$S_{\frac{1}{2}}\varphi_j(x, y, z) = e^{-E(\frac{1}{2})}\psi_j(x, y, z)$$

we have

$$\left.\begin{aligned} S_{\frac{1}{2}}\psi_j(x, y, z) &= (-i)^j e^{-E(\frac{1}{2})}\varphi_j(x, y, z) \\ S_{\frac{2r+1}{2}}\psi_j(x, y, z) &= (-i)^{j(r+1)} e^{-E(r+\frac{1}{2})}\varphi_j(x, y, z) \end{aligned}\right\} \tag{10}$$

In general, the effect of S_1 is to change $E(m)$ into $E(m+1)$, while S_p changes $E(m)$ into $E(m+p)$. Hence the effect of S_p on our functions is the same as that of S^p, to within an exponential factor which may be taken out from under the sign of summation. Similarly $S_{\frac{1}{2}}$ changes $E(m)$ into $E(m+\frac{1}{2})$, and $S_{\frac{2r+1}{2}}$ changes $E(m)$ into $E(m+r+\frac{1}{2})$. Hence we see here also that, to within an exponential factor as above, the effect of S_1 is the same as that of $S^2_{\frac{1}{2}}$, and finally, that of $S_{\frac{2r+1}{2}}$ the same as that of $S^{2r+1}_{\frac{1}{2}}$, or of $S_1^r S_{\frac{1}{2}}$, or of $S_r S_{\frac{1}{2}}$.

Changing x into $-x$ and z into $-z$ simultaneously has the effect of changing m into $-m$ in φ, and m into $-m-1$ in ψ; hence it

leaves φ_0, φ_2 and ψ_0 unaltered,
interchanges φ_1 and φ_3,
changes ψ_1 into $-i\psi_3$,
ψ_3 into $i\psi_1$, and
ψ_2 into $-\psi_2$.

Consequently, as regards x and z simultaneously, we see that

φ_0, φ_2, ψ_0, $\varphi_1\varphi_3$, $\psi_1\psi_3$ are even, and
ψ_2 is odd.

Changing the sign of x or of z alone, or of y changes the values of all the functions in such a way that no conclusions as to parity can be drawn in these cases.

In the case of each of these functions we may find those zeros which, like the zeros of the Θ-functions, cause the vanishing of the function by the cancellation in pairs of the terms of the series defining it. This we can accomplish by the examination of the function $\psi_2(x, y, z)$. We have, in fact,

$$\psi_2(0, y, 0) = \sum_m (-1)^m e^{a(m+\frac{1}{2})^4 + 2\pi i \frac{y}{\omega_2}(m+\frac{1}{2})^2}.$$

Changing m into $-m-1$ we get

$$\psi_2(0, y, 0) = \sum_m (-1)^{-m-1} e^{a(m+\frac{1}{2})^4 + 2\pi i \frac{y}{\omega_2}(m+\frac{1}{2})^2}$$
$$= -\sum_m (-1)^m e^{a(m+\frac{1}{2})^4 + 2\pi i \frac{y}{\omega_2}(m+\frac{1}{2})^2}.$$

Hence

$$\psi_2(0, y, 0) = 0.$$

Or, from (8′), we have more generally

$$\psi_2(4h\omega\ , y, \ l\omega_3) = 0$$

where h and l are any integers and y is anything at all.

Finally, applying the substitution S_q, $\psi_2(x, y, z)$ is reproduced multiplied by the finite factor $(-1)^q e^{-E(q)}$ which is different from zero. Hence we have $\psi_2(4h\omega_1 + \frac{2a\omega_1}{\pi i} q, y + 12h\omega_2 q + \frac{3a\omega_2}{\pi i} q^2, l\omega_3 + \frac{2y\omega_3}{\omega_2} q + 12h\omega_3 q^2 + \frac{2a\omega_3}{\pi i} q^3) = 0$. The most general set of zeros of ψ_2 is then, without loss of generality, from (8′),

$$\left.\begin{aligned} x &= 4h\omega_1 + \frac{2\omega_1 a}{\pi i} q \\ y &= y + 2k\omega_2 + \frac{3\omega_2 a}{\pi i} q^2 \\ z &= l\omega_3 + \frac{2\omega_3 y}{\omega_2} q + \frac{2\omega_3 a}{\pi i} q^3 \end{aligned}\right\} \qquad (11)$$

From the second and fifth equations of (7) and the second equation of (10) we can write the following table of zeros where, as above,

$h, k, l, q,$ are any integers,
y anything whatever.

Zeros of	$x=$	$y=$	$z=$
ψ_0	$(4h+2)\,\omega_1+\frac{2a\omega_1}{\pi i}q$	$y+2k\omega_2+\frac{3a\omega_2}{\pi i}q^2$	$l\omega_3+2y\frac{\omega_3}{\omega_2}q+\frac{2a\omega_3}{\pi i}q^3$
	or		
	$4h\omega_1+\frac{2a\omega_1}{\pi i}q$	$y+2k\omega_2+\frac{3a\omega_2}{\pi i}q^2$	$(l+\frac{1}{2})\,\omega_3+2y\frac{\omega_3}{\omega_2}q+\frac{2a\omega_3}{\pi i}q^3$
ψ_1	$(4h+1)\,\omega_1+\frac{2a\omega_1}{\pi i}q$	$y+2k\omega_2+\frac{3a\omega_2}{\pi i}q^2$	$l\omega_3+2y\frac{\omega_3}{\omega_2}q+\frac{2a\omega_3}{\pi i}q^3$
	or		
	$4h\omega_1+\frac{2a\omega_1}{\pi i}q$	$y+2k\omega_2+\frac{3a\omega_2}{\pi i}q^2$	$(l+\frac{1}{4})\,\omega_3+2y\frac{\omega_3}{\omega_2}q+\frac{2a\omega_3}{\pi i}q^3$
ψ_2	$4h\omega_1+\frac{2a\omega_1}{\pi i}q$	$y+2k\omega_2+\frac{3a\omega_2}{\pi i}q^2$	$l\omega_3+2y\frac{\omega_3}{\omega_2}q+\frac{2a\omega_3}{\pi i}q^3$
ψ_3	$(4h+3)\,\omega_1+\frac{2a\omega_1}{\pi i}q$	$y+2k\omega_2+\frac{3a\omega_2}{\pi i}q^2$	$l\omega_3+2y\frac{\omega_3}{\omega_2}q+\frac{2a\omega_3}{\pi i}q^3$
	or		
	$4h\omega_1+\frac{2a\omega_1}{\pi i}q$	$y+2k\omega_2+\frac{3a\omega_2}{\pi i}q^2$	$(l+\frac{3}{4})\,\omega_3+2y\frac{\omega_3}{\omega_2}q+\frac{2a\omega_3}{\pi i}q^3$
φ_0	$h\omega_1+\frac{2a\omega_1}{\pi i}\left(\frac{2r+1}{2}\right)$	$y+k\omega_2+\frac{3a\omega_2}{\pi i}\left(\frac{2r+1}{2}\right)^2$	$(l+\frac{1}{2})\,\omega_3+2y\frac{\omega_3}{\omega_2}\left(\frac{2r+1}{2}\right)+\frac{2a\omega_3}{\pi i}\left(\frac{2r+1}{2}\right)^3$
φ_1	$h\omega_1+\frac{2a\omega_1}{\pi i}\left(\frac{2r+1}{2}\right)$	$y+k\omega_2+\frac{3a\omega_2}{\pi i}\left(\frac{2r+1}{2}\right)^2$	$(l+\frac{1}{4})\,\omega_3+2y\frac{\omega_3}{\omega_2}\left(\frac{2r+1}{2}\right)+\frac{2a\omega_3}{\pi i}\left(\frac{2r+1}{2}\right)^3$
φ_2	$h\omega_1+\frac{2a\omega_1}{\pi i}\left(\frac{2r+1}{2}\right)$	$y+k\omega_2+\frac{3a\omega_2}{\pi i}\left(\frac{2r+1}{2}\right)^2$	$l\omega_3+2y\frac{\omega_3}{\omega_2}\left(\frac{2r+1}{2}\right)+\frac{2a\omega_3}{\pi i}\left(\frac{2r+1}{2}\right)^3$
φ_3	$h\omega_1+\frac{2a\omega_1}{\pi i}\left(\frac{2r+1}{2}\right)$	$y+k\omega_2+\frac{3a\omega_2}{\pi i}\left(\frac{2r+1}{2}\right)^2$	$(l+\frac{3}{4})\,\omega_3+2y\frac{\omega_3}{\omega_2}\left(\frac{2r+1}{2}\right)+\frac{2a\omega_3}{\pi i}\left(\frac{2r+1}{2}\right)^3$

Putting $h = k = l = q = r = 0$ we get the following simple zeros:

Zeros of	$x =$	$y =$	$z = 0$
ψ_0	$2\omega_1$ or 0	y y	0 $\frac{\omega_3}{2}$
ψ_1	ω_1 or 0	y y	0 $\frac{\omega_3}{4}$
ψ_2	0	y	0
ψ_3	$3\omega_1$ or 0	y y	0 $\frac{3\omega_3}{4}$
φ_0	$\frac{a\omega_1}{\pi i}$	$y + \frac{3a\omega_2}{4\pi i}$	$\frac{\omega_3}{2} + \frac{a\omega_3}{4\pi i}$
φ_1	$\frac{a\omega_1}{\pi i}$	$y + \frac{3a\omega_2}{4\pi i}$	$\frac{\omega_3}{4} + \frac{a\omega_3}{4\pi i}$
φ_2	$\frac{a\omega_1}{\pi i}$	$y + \frac{3a\omega_2}{4\pi i}$	$\frac{a\omega_3}{4\pi i}$
φ_3	$\frac{a\omega_1}{\pi i}$	$y + \frac{3a\omega_2}{4\pi i}$	$\frac{3\omega_3}{4} + \frac{a\omega_3}{4\pi i}$

The zeros of $\varphi(x, y, z)$ might have been gotten directly, as follows:

$$\varphi_0(x, y, z) = \sum_m e^{am^4 + 2\pi i \left(\frac{x}{\omega_1} m^3 + \frac{y}{\omega_2} m^2 + \frac{z}{\omega_3} m\right)}$$

$$= \sum_m e^{a(\mu - m)^4 + 2\pi i \left[\frac{x}{\omega_1}(\mu - m)^3 + \frac{y}{\omega_2}(\mu - m)^2 + \frac{z}{\omega_3}(\mu - m)\right]}.$$

The corresponding terms of these two series will be equal but of opposite sign for those values of x, y, z which make the exponents of e in the two cases differ by an odd multiple of πi for all values of m. Such values of x, y, z will evidently cause $\varphi_0(x, y, z)$ to vanish. We are to have, then

$$a(\mu - m)^4 + 2\pi i \left[\frac{x}{\omega_1}(\mu - m)^3 + \frac{y}{\omega_2}(\mu - m)^2 + \frac{z}{\omega_3}(\mu - m)\right]$$
$$- \left[am^4 + 2\pi i\left(\frac{x}{\omega_1} m^3 + \frac{y}{\omega_2} m^2 + \frac{z}{\omega_3} m\right)\right] = (2k+1)\pi i,$$

which on reduction becomes

$$(\mu - 2m)\left\{\left(\frac{\pi i x}{\omega_1} + a\mu\right)(2m^2 - 2m\mu + \mu + \mu^2) + \left(\frac{\pi i x}{\omega_1}\mu^2 + \frac{2\pi i y}{\omega_2}\mu + \frac{2\pi i z}{\omega_3}\right)\right\} = (2k+1)\pi i.$$

This condition is satisfied if

$$\left.\begin{aligned} &\mu \text{ is an odd integer,} \\ &\frac{\pi i x}{\omega_1} + a\mu = \frac{\lambda \pi i}{2}, \\ &\frac{\pi i x}{\omega_1}\mu^2 + \frac{2\pi i y}{\omega_2}\mu + \frac{2\pi i z}{\omega_3} = [\mu^2\lambda + 2(\lambda + 2\rho + 1)]\frac{\pi i}{2} \end{aligned}\right\} \qquad (12)$$

where λ and ρ are any integers.

To obtain the zeros of $\varphi_0(x, y, z)$ we need only put

$$\left.\begin{aligned} &x = \left(\frac{\lambda}{2} - \frac{a\mu}{\pi i}\right)\omega_1 \\ &y \quad \text{anything, as before.} \\ &z = \left[\frac{\lambda + 2\rho + 1}{2} - \frac{\mu y}{\omega_2} + \frac{a\mu^3}{2\pi i}\right]\omega_3 \end{aligned}\right\} \qquad (13)$$

This can be readily verified, for, putting these values in $\varphi(x, y, z)$, we have

$$\sum_{m=-\infty}^{m=\infty} e^{am^4 + 2\pi i\left[\left(\frac{\lambda}{2} - \frac{a\mu}{\pi i}\right)m^3 + \frac{y}{\omega_2}m^2 + \left(\frac{\lambda + 2\rho + 1}{2} - \frac{\mu y}{\omega_2} + \frac{a\mu^3}{2\pi i}\right)m\right]}$$
$$= \sum e^{a(m^4 - 2m^3\mu + m\mu^3) + 2\pi i \frac{y}{\omega_2} m(m - \mu) + 2\pi i\left(\frac{\lambda}{2}m^3 + \frac{\lambda + 2\rho + 1}{2}m\right)}$$
$$= e^{-\frac{a\mu^4}{16}} \sum (-1)^m e^{a\left(m - \frac{\mu}{2}\right)^4 - \frac{3}{2}a(m - \mu)\mu^2 m + 2\pi i \frac{y}{\omega_2}(m - \mu)m}$$

because

$$e^{\pi i(\lambda m^2 + \lambda + 2\rho + 1)m} = (-1)^m,$$

since for λ even, $\lambda m^2 + \lambda + 2\rho + 1$ is odd

λ odd, $\lambda m^2 + \lambda + 2\rho + 1$ is even or odd according as m is even or odd.

When m is replaced by $\mu - m$, the expression above is only altered by having $(-1)^m$ replaced by $(-1)^{\mu - m}$. Since μ is an odd integer,

$$(-1)^{\mu - m} = -(-1)^m,$$

i. e. for the above values of the variables, the function is equal to its negative, and is hence equal to zero.

In the above it was stated that y may be taken arbitrary. As a matter of fact, either y or z may be so chosen, since these two variables are, from (12), subjected only to the one condition

$$\frac{2\pi i}{\omega_2} y\mu + \frac{2\pi i}{\omega_3} z = (\lambda + 2\rho + 1)\pi i + a\mu^3. \tag{14}$$

If z be taken arbitrary, our zeros will be

$$\left.\begin{aligned} x &= \left(\frac{\lambda}{2} - \frac{a\mu}{\pi i}\right)\omega_1 \\ y &= \left(\frac{\lambda + 2\rho + 1}{2\mu} + \frac{a\mu^2}{2\pi i} - \frac{z}{\mu\omega_3}\right)\omega_2 \\ z &= \text{anything} \end{aligned}\right\} \tag{15}$$

For, on substituting these values, we get

$$\sum_m e^{am^4 + 2\pi i \left[\left(\frac{\lambda}{2} - \frac{a\mu}{\pi i}\right)m^3 + \left(\frac{\lambda + 2\rho + 1}{2\mu} + \frac{a\mu^2}{2\pi i} - \frac{z}{\mu\omega_3}\right)m^2 + \frac{z}{\omega_3} m\right]}$$

$$= e^{-\frac{a\mu^4}{16}} \sum e^{a\left(m - \frac{\mu}{2}\right)^4 - \frac{a}{2}\mu^2 m(m - \mu) - \frac{2\pi i}{\mu\omega_3} z(m^2 - \mu) + \pi i\lambda m^3 + m^2 \pi i \frac{\lambda + 2\rho + 1}{\mu}}.$$

The effect of changing m into $\mu - m$ is to replace the factor $e^{\lambda m^3 \pi i}$ in the above by

$$e^{[\lambda(\mu - m)^3 + (\lambda + 2\rho + 1)\mu - 2(\lambda + 2\rho + 1)m]\pi i}$$

which may be written, for brevity, $e^{L\pi i}$.

If λ is even, L is odd

λ is odd, $e^{\lambda m^3 \pi i} = (-1)^m$, and $e^{L\pi i} = (-1)^{\mu - m}$.

So that for all values of λ

$$e^{L\pi i} = -e^{\lambda m^3 \pi i}.$$

Which shows us, in the same way as before, that (15) is also a set of zeros.

The fact that z in (13) and in (11) contained the arbitrary quantity y might have also assured us that we could so choose y as to give z any value we please, and still have the resulting value of the function zero.

The zeros of φ_1, φ_2, φ_3 and those of ψ_0, ψ_1, ψ_2, ψ_3 can also be calculated directly in the same way we have just found those of $\varphi_0(x, y, z)$, or they can be derived from those of φ_0 in a manner similar to that used in obtaining the zeros of all the rest from those of ψ_2.

A comparison of our set of zeros for φ_0 obtained by the two methods, which, in fact however, are the same in principle, will manifestly show them to be identical, if account be taken of (3).

By the first method, the simplest zeros were first obtained, and from these we determined the most general zeros, by observing what operations could be performed upon the function without altering its value, except, perhaps, as to a finite factor different from zero. By the second method we obtain the most general set of zeros at once. The simplest zeros are then gotten by putting

$$\lambda = \rho = 0 \qquad \mu = -1 .$$

I say that (13), for example, is the most general set of zeros possible of the kind here considered, for all the operations which leave the value of the function unaltered, or unaltered except as to a finite factor other than zero, are there provided for.

Thus λ so enters, that a change in it by an even integer amount corresponds to a change in x and z by some multiple of ω_1 and ω_3 respectively; while the change in λ will be an odd integer when x and z are increased or diminished by the same odd multiple of $\frac{\omega_1}{2}$ and $\frac{\omega_3}{2}$ respectively, which by (3) does not alter the value of the function.

The presence of ρ permits a change in z alone by any multiple of ω_3.

y so enters in the value of z that if changed by an integer multiple of ω_2, z will be changed by an integer multiple of ω_3, and if altered by an odd multiple of $\frac{\omega_2}{2}$, the effect on z will be to change it by an odd multiple of $\frac{\omega_3}{2}$.

Finally, μ being an odd integer, it represents the result of the operation of S_p, where p is any integer, the effect of which is, as we saw, to leave the value of the function unaltered except as to a finite factor other than zero.

In the zeros we have found, namely those which cause the cancellation, in pairs, of the terms of the series, only y or z (but not both simultaneously) may be taken arbitrary, while x cannot be taken arbitrary at all, since it enters alone in one of the equations of condition (12). No way of discovering other zeros has suggested itself; and it remains a question whether other zeros exist or all the zeros are confined within the above restrictions.

III.

The quotients

$$\frac{\varphi_0}{\varphi_2},\quad \frac{\varphi_1}{\varphi_2},\quad \frac{\varphi_3}{\varphi_2},\quad \frac{\psi_0}{\varphi_2},\quad \frac{\psi_1}{\varphi_2},\quad \frac{\psi_2}{\varphi_2},\quad \frac{\psi_3}{\varphi_2}$$

are doubly periodic functions, the substitutions

S_2 and	$(x, y, z;\ x+\alpha\omega_1,\ y+\beta\omega_2,\ z+\gamma\omega_3)$	leaving	$\frac{\varphi_0}{\varphi_2}$	unaltered.
S_4 "	$(x, y, z;\ x+\alpha\omega_1,\ y+\beta\omega_2,\ z+\gamma\omega_3)$	"	$\frac{\varphi_1}{\varphi_2}$ and $\frac{\varphi_3}{\varphi_2}$	"
S_2 "	$(x, y, z;\ x+8\alpha\omega_1,\ y+4\beta\omega_2,\ z+2\gamma\omega_3)$	"	$\frac{\psi_0}{\varphi_2}$	"
S_1 "	$(x, y, z;\ x+8\alpha\omega_1,\ y+4\beta\omega_2,\ z+2\gamma\omega_3)$	"	$\frac{\psi_2}{\varphi_2}$	"
S_4 "	$(x, y, z;\ x+8\alpha\omega_1,\ y+4\beta\omega_2,\ z+2\gamma\omega_3)$	"	$\frac{\psi_1}{\varphi_2}$ and $\frac{\psi_3}{\varphi^2}$	"

α, β, γ being any integers.

If now we turn our attention to the derivatives of these quotients with respect to x, y or z and inquire whether, analogously to the elliptic functions, these derivatives are expressible in terms of any combination of the quotients themselves, it would seem that such is not the case.

We shall first consider the derivatives with respect to z. For convenience of reference, the following tables may be of service.

We saw, page 13, that

$\varphi_0,\ \varphi_2,\ \psi_0,\ \varphi_1\varphi_3,\ \psi_1\psi_3$ are even as to x and z jointly, and that
ψ_2 is odd.

Consequently

$$\frac{\partial\varphi_0}{\partial z},\ \frac{\partial\varphi_2}{\partial z},\ \frac{\partial\psi_0}{\partial z},\ \frac{\partial(\varphi_1\varphi_3)}{\partial z},\ \frac{\partial(\psi_1\psi_3)}{\partial z}\ \text{are odd, and}$$

$$\frac{\partial\psi_2}{\partial z}\quad \text{is even.}$$

$$\frac{\partial}{\partial z}\varphi_j(x+\omega_1, y, z)=\frac{\partial}{\partial z}\varphi_j(x, y+\omega_2, z)=\frac{\partial}{\partial z}\varphi_j(x, y, z+\omega_3)=\frac{\partial}{\partial z}\varphi_j(x, y, z)$$

$$S_1\ \frac{\partial\varphi_j}{\partial z}=(-i)^j e^{-E(1)}\left[\frac{\partial\varphi_j}{\partial z}-\frac{2\pi i}{\omega_3}\varphi_j\right],\quad S_{\frac{1}{2}}\ \frac{\partial\varphi_j}{\partial z}= e^{-E(\frac{1}{2})}\left[\frac{\partial\psi_j}{\partial z}-\frac{\pi i}{\omega_3}\psi_j\right],$$

$$S_1\ \frac{\partial\psi_j}{\partial z}=(-i)^j e^{-E(1)}\left[\frac{\partial\psi_j}{\partial z}-\frac{2\pi i}{\omega_3}\psi_j\right],\quad S_{\frac{1}{2}}\ \frac{\partial\psi_j}{\partial z}=(-i)^j e^{-E(\frac{1}{2})}\left[\frac{\partial\varphi_j}{\partial z}-\frac{\pi i}{\omega_3}\varphi_j\right],$$

$$j=0,\ 1,\ 2,\ 3.$$

The effect of adding multiples of ω_1, ω_2 and ω_3 to x, y and z respectively in $\dfrac{\partial\psi_j}{\partial z}$ is the same as given in (7) in the case of ψ_j.

Let us now consider

$$\frac{\partial}{\partial z}\left(\frac{\psi_2}{\varphi_2}\right)=\frac{\varphi_2\dfrac{\partial}{\partial z}\psi_2-\psi_2\dfrac{\partial}{\partial z}\varphi_2}{\varphi_2^2}.$$

If the numerator of the second member is expressible rationally in terms of φ_j and ψ_k $(j,\ k=0,\ 1,\ 2,\ 3)$ it will be a linear combination of quadratic functions of these, for it has no infinities for finite values of the variables, and the effect of the substitution S_1 is to reproduce it multiplied by $e^{-2E(1)}$. Moreover it is even as to x and z jointly, and is changed in sign when z is changed into $z+\omega_3$. Finally it is reproduced multiplied by the factor $e^{-2E(\frac{1}{2})}$ when operated on by $S_{\frac{1}{2}}$.

Of all the 36 combinations of φ_j and ψ_k taken two at a time, only one, $\varphi_0\,\psi_0$ satisfies all these requirements. Besides

$$\varphi_2\frac{\partial\psi_2}{\partial z}-\psi_2\frac{\partial\varphi_2}{\partial z}$$

is zero whenever φ_0 and whenever ψ_0 is. Hence it would seem that we could write

$$\varphi_2\frac{\partial\psi_2}{\partial z}-\psi_2\frac{\partial\varphi_2}{\partial z}=A\varphi_0\,\psi_0$$

where, since this relation must hold when operated on by S_1 and when x, y, z are altered by multiples of ω_1, ω_2, ω_3 respectively, A is a constant and equal to

$$\frac{\varphi_2(0,\ 0,\ 0)\dfrac{\partial}{\partial z}\psi_2(0,\ 0,\ 0)}{\varphi_0(0,\ 0,\ 0)\quad\psi_0(0,\ 0,\ 0)}.$$

So that we have finally

$$\frac{\partial}{\partial z}\left[\frac{\psi_2(x,\ y,\ z)}{\varphi_2(x,\ y,\ z)}\right]=\frac{\varphi_2(0,\ 0,\ 0)\dfrac{\partial}{\partial z}\psi_2(0,\ 0,\ 0)}{\varphi_0(0,\ 0,\ 0)\quad\psi_0(0,\ 0,\ 0)}\cdot\frac{\varphi_0(x,\ y,\ z)\ \ \psi_0(x,\ y,\ z)}{\varphi_2^2(x,\ y,\ z)}. \tag{1}$$

If now we apply the substitution

$$(x,\ y,\ z;\ x+\omega_1,\ y,\ z)$$

we shall get

$$\frac{\partial}{\partial z}\left[\frac{\psi_3(x,\ y,\ z)}{\varphi_2(x,\ y,\ z)}\right]=\frac{\varphi_2(0,\ 0,\ 0)\dfrac{\partial}{\partial z}\psi_2(0,\ 0,\ 0)}{\varphi_0(0,\ 0,\ 0)\quad\psi_0(0,\ 0,\ 0)}\cdot\frac{\varphi_0(x,\ y,\ z)\quad\psi_1(x,\ y,\ z)}{\varphi_2^2(x,\ y,\ z)}$$

which must hold for all values of x, y, z. Substituting the first set of zeros for ψ_3 given on page 16, this equation is satisfied. But using the second set we get

$$\frac{\varphi_2(0,0,0)}{\varphi_0(0,0,0)} \frac{\frac{\partial}{\partial z}\psi_2(0,0,0)}{\psi_0(0,0,0)} = \frac{\varphi_1(0,0,0)}{\varphi_3(0,0,0)} \frac{\frac{\partial}{\partial z}\psi_2(0,0,0)}{\psi_0(0,0,0)}$$

which is manifestly not true, for

$$\varphi_1(0,0,0) = \varphi_3(0,0,0)$$

while

$$\varphi_0(0,0,0) \neq \varphi_2(0,0,0)$$

as may be seen from the definition of these functions. Hence we must conclude that a relation of the form (1) does not exist.

Again, if we consider the numerator of

$$\frac{\partial}{\partial z}\left(\frac{\psi_0}{\varphi_2}\right) = \frac{\varphi_2 \frac{\partial \psi_0}{\partial z} - \psi_0 \frac{\partial \varphi_2}{\partial z}}{\varphi_2^2}$$

it will be found that of all the combinations of φ_j and ψ_j only $\varphi_0\,\psi_2$ satisfies all the conditions that it does. Moreover, the numerator vanishes for all our zeros of φ_0 and of ψ_2. But on writing

$$\frac{\partial}{\partial z}\left[\frac{\psi_0(x,y,z)}{\varphi_2(x,y,z)}\right] = B\,\frac{\varphi_0(x,y,z)\quad \psi_2(x,y,z)}{\varphi_2^2(x,y,z)}$$

where, as before, B can only be a constant, we shall find, on substituting the first set of zeros for ψ_0,

$$B = \frac{\varphi_2(0,0,0)}{\varphi_0(0,0,0)} \frac{\frac{\partial}{\partial z}\psi_2(0,0,0)}{\psi_0(0,0,0)}.$$

Using the second set of zeros for ψ_0, we get

$$B = \frac{\varphi_0(0,0,0)}{\varphi_2(0,0,0)} \frac{\frac{\partial}{\partial z}\psi_2(0,0,0)}{\psi_0(0,0,0)}.$$

These two values are not the same, since

$$\varphi_2^2(0,0,0) \neq \varphi_0^2(0,0,0).$$

Hence we conclude that no relation of the type (3) exists. In the same way, the expressions for the derivatives of all the various quotients of a ψ by a φ in terms of the quotients themselves break down.

The difficulty seems to lie in the fact that the fundamental periods of ψ_j are multiples of those of φ_j. For we can pass from ψ_j to ψ_{j+1} by either of the two substitutions

$$(x, y, z; x+\omega_1, y, z), \quad (x, y, z; x, y, z+\frac{\omega_3}{4}).$$

At the same time φ_j from which ψ_j was derived as the result of the substitution $S_{\frac{1}{2}}$, is left unaltered by the first of these substitutions, and is changed into φ_{j+1} by the second.

All our relations, arrived at in a manner similar to that indicated above, broke down when subjected to this test.

It should be noted that this test does not apply in the case of the Θ- and H-functions. As no general theorem analogous to that made use of in this connection in the case of the Θ-functions could be established for the functions here considered, the above method was employed, to show that no such relations exist.

In a similar way, no quadratic relations between any of the φ_j and ψ_k satisfying all the tests at our command, could be found.

The objection, above mentioned, as holding good against the relations between the derivatives of the quotients and the quotients themselves, do not seem to hold in the following cases, where only the quotients of the φ_j or of the ψ_k are involved separately. Thus, it was found that

$$\begin{aligned}\frac{\partial}{\partial z}\left[\frac{\varphi_0(x, y, z)}{\varphi_2(x, y, z)}\right] &= \frac{\varphi_2(x, y, z)\frac{\partial}{\partial z}\varphi_0(x, y, z)-\varphi_0(x, y, z)\frac{\partial}{\partial z}\varphi_2(x, y, z)}{\varphi_2^2(x, y, z)} \\ &= \frac{\psi_0(0, 0, 0)\frac{\partial}{\partial z}\psi_2(0, 0, 0)}{\psi_3^2(0, 0, 0)-\psi_1^2(0, 0, 0)}\cdot\frac{\varphi_1^2(x, y, z)-\varphi_3^2(x, y, z)}{\varphi_2^2(x, y, z)} \qquad (4)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial z}\left[\frac{\psi_0(x, y, z)}{\psi_2(x, y, z)}\right] &= \frac{\psi_2(x, y, z)\frac{\partial}{\partial z}\psi_0(x, y, z)-\psi_0(x, y, z)\frac{\partial}{\partial z}\psi_2(x, y, z)}{\psi_2^2(x, y, z)} \\ &= \frac{\psi_0(0, 0, 0)\frac{\partial}{\partial z}\psi_2(0, 0, 0)}{\psi_3^2(0, 0, 0)-\psi_1^2(0, 0, 0)}\cdot\frac{\psi_1^2(x, y, z)-\psi_3^2(x, y, z)}{\varphi_2^2(x, y, z)}, \qquad (5)\end{aligned}$$

which may be derived from (4) as the result of operating with $S_{\frac{1}{2}}$, together with the other relations derived from them by all the operations at our command, satisfy all the tests that were applied to them.

These results are given here, not as proved relations, but as such which have not been disproved.

The derivatives with respect to x and y are more complicated than those with respect to z. Hence it will be at least no easier to establish relations involving the former than to establish any involving the latter. Thus

$$\frac{\partial\varphi_0}{\partial y} = \frac{2\pi i}{\omega_2} \sum_{m=-\infty}^{m=\infty} m^2\, e^{E(m)}$$

$$\begin{aligned} S_1 \frac{\partial\varphi_0}{\partial y} &= \frac{2\pi i}{\omega_2}\, e^{-E(1)} \sum_{m=-\infty}^{m=\infty} m^2\, e^{E(m+1)} \\ &= \frac{2\pi i}{\omega_2}\, e^{-E(1)} \sum_{m=-\infty}^{m=\infty} (m-1)^2\, e^{E(m)} \\ &= e^{-E(1)} \left[\frac{2\pi i}{\omega_2} \sum m^2\, e^{E(m)} - \frac{4\pi i}{\omega_2} \sum m e^{E(m)} + \frac{2\pi i}{\omega_2} \sum e^{E(m)} \right] \\ &= e^{-E(1)} \left[\frac{\partial\varphi_0}{\partial y} - \frac{2\omega_3}{\omega_2} \frac{\partial\varphi_0}{\partial z} + \frac{2\pi i}{\omega_2} \varphi_0 \right]. \end{aligned}$$

Similarly

$$S_1 \frac{\partial\varphi_0}{\partial x} = e^{-E(1)} \left[\frac{\partial\varphi_0}{\partial x} - 3\,\frac{\omega_2}{\omega_1} \frac{\partial\varphi_0}{\partial y} + \frac{3\omega_3}{\omega_1} \frac{\partial\varphi_0}{\partial z} - \frac{2\pi i}{\omega_1} \varphi_0 \right].$$

In some respects the functions

$$\Phi_0 = \varphi_0\, \varphi_2 \qquad \Phi_1 = \varphi_1\, \varphi_3 \qquad \Psi_0 = \psi_0\, \psi_2 \qquad \Psi_1 = \psi_1\, \psi_3$$

are simpler than the φ_j and ψ_k. Thus, from what was seen before,

Φ_0, Φ_1, Ψ_1 are even as to x and z simultaneously, and
Ψ_0 is odd.

$$\begin{aligned} S_1\, \Phi_0 &= -\, e^{-2E(1)}\, \Phi_0 & S_1\, \Psi_0 &= -\, e^{-2E(1)}\, \Psi_0 \\ S_1\, \Phi_1 &= e^{-2E(1)}\, \Phi_1 & S_1\, \Psi_1 &= e^{-2E(1)}\, \Psi_1 \\ S_{\frac{1}{2}}\, \Phi_0 &= e^{-2E(\frac{1}{2})}\, \Psi_0 & S_{\frac{1}{2}}\, \Psi_0 &= -\, e^{-2E(\frac{1}{2})}\, \Phi_0 \\ S_{\frac{1}{2}}\, \Phi_1 &= e^{-2E(\frac{1}{2})}\, \Psi_1 & S_{\frac{1}{2}}\, \Psi_1 &= e^{-2E(\frac{1}{2})}\, \Phi_1 \end{aligned}$$

$$\left.\begin{aligned} \Phi_j(x+\omega_1,\ y+\omega_2,\ z+\omega_3) &= \Phi_j(x,\ y,\ z) \\ \Phi_j(x+\omega_1,\ y,\ z) &= \Phi_j(x,\ y,\ z) \\ \Phi_j(x,\ y+\omega_2,\ z) &= \Phi_j(x,\ y,\ z) \\ \Phi_j(x,\ y,\ z+\omega_3) &= \Phi_j(x,\ y,\ z) \end{aligned}\right\} \quad j = 0,\ 1$$

$$\left.\begin{aligned} \Psi_j(x+\omega_1,\ y+\omega_2,\ z+\omega_3) &= -\, i\, \Psi_{j+1}(x,\ y,\ z) \\ \Psi_j(x+\omega_1,\ y,\ z) &= i\, \Psi_{j+1}(x,\ y,\ z) \\ \Psi_j(x,\ y+\omega_2,\ z) &= -\, \Psi_j(x,\ y,\ z) \\ \Psi_j(x,\ y,\ z+\omega_3) &= \Psi_j(x,\ y,\ z) \\ \Psi_j(x,\ y,\ z+\tfrac{\omega_3}{4}) &= i\, \Psi_{j+1}(x,\ y,\ z) \end{aligned}\right\} \quad \begin{aligned} j &= 0,\ 1 \\ \Psi_2 &= \Psi_0 \end{aligned}$$

while, as before

$$\Psi_j\left(x+\frac{\omega_1}{2},\ y,\ z\right) \text{ and } \Psi_j\left(x+\frac{\omega_1}{4},\ y,\ z\right)$$

are entirely new functions, not expressible in terms of $\Phi_0,\ \Phi_1,\ \Psi_0,\ \Psi_1$.

$$\frac{\partial \Phi_0}{\partial z}=\varphi_0\frac{\partial \varphi_2}{\partial z}+\varphi_2\frac{\partial \varphi_0}{\partial z}$$

$$S_1\frac{\partial \Phi_0}{\partial z}=-e^{-2E(1)}\varphi_0\left[\frac{\partial \varphi_2}{\partial z}-\frac{2\pi i}{\omega_3}\varphi_2\right]-e^{-2E(1)}\varphi_2\left[\frac{\partial \varphi_0}{\partial z}-\frac{2\pi i}{\omega_3}\varphi_0\right]$$

$$=-e^{-2E(1)}\left[\varphi_0\frac{\partial \varphi_2}{\partial z}+\varphi_2\frac{\partial \varphi_0}{\partial z}\right]+\frac{4\pi i}{\omega_3}e^{-2E(1)}\varphi_0\,\varphi_2$$

$$\therefore\quad S_1\frac{\partial \Phi_0}{\partial z}=-e^{-2E(1)}\left[\frac{\partial \Phi_0}{\partial z}-\frac{4\pi i}{\omega_3}\Phi_0\right].$$

Similarly

$$S_1\frac{\partial \Phi_1}{\partial z}=\quad e^{-2E(1)}\left[\frac{\partial \Phi_1}{\partial z}-\frac{4\pi i}{\omega_3}\Phi_1\right]$$

$$S_1\frac{\partial \Psi_0}{\partial z}=-e^{-2E(1)}\left[\frac{\partial \Psi_0}{\partial z}-\frac{4\pi i}{\omega_3}\Psi_0\right]$$

$$S_1\frac{\partial \Psi_1}{\partial z}=\quad e^{-2E(1)}\left[\frac{\partial \Psi_1}{\partial z}-\frac{4\pi i}{\omega_3}\Psi_1\right]$$

and

$$S_{\frac{1}{2}}\frac{\partial \Phi_0}{\partial z}=\quad e^{-2E(\frac{1}{2})}\left[\frac{\partial \Psi_0}{\partial z}-\frac{2\pi i}{\omega_3}\Psi_0\right]$$

$$S_{\frac{1}{2}}\frac{\partial \Phi_1}{\partial z}=\quad e^{-2E(\frac{1}{2})}\left[\frac{\partial \Psi_1}{\partial z}-\frac{2\pi i}{\omega_3}\Psi_1\right]$$

$$S_{\frac{1}{2}}\frac{\partial \Psi_0}{\partial z}=-e^{-2E(\frac{1}{2})}\left[\frac{\partial \Phi_0}{\partial z}-\frac{2\pi i}{\omega_3}\Phi_0\right]$$

$$S_{\frac{1}{2}}\frac{\partial \Psi_1}{\partial z}=\quad e^{-2E(\frac{1}{2})}\left[\frac{\partial \Phi_1}{\partial z}-\frac{2\pi i}{\omega_3}\Phi_1\right].$$

If by aid of these formulæ, we attempt to express the derivative with respect to z of the quotient of any two of our functions $\Phi_0,\ \Phi_1,\ \Psi_0,\ \Psi_1$ in terms of any or all of the quotients, we will meet with the same difficulties as before. Thus if we take, for example

$$\frac{\partial}{\partial z}\left(\frac{\Psi_0}{\Phi_0}\right)=\frac{\Phi_0\dfrac{\partial \Psi_0}{\partial z}-\Psi_0\dfrac{\partial \Phi_0}{\partial z}}{\Phi_0^2}$$

we shall find that of all the combinations of our functions only $\Phi_1\,\Psi_1$ behaves exactly like $\Phi_0\dfrac{\partial \Psi_0}{\partial z}-\Psi_0\dfrac{\partial \Phi_0}{\partial z}$ when put to the test of all the above operations, and besides, the numerator of our expression for the derivative vanishes for all the known zeros of Φ_1 and of Ψ_1. But on writing

$$\frac{\partial}{\partial z}\left[\frac{\Psi_0(x,\ y,\ z)}{\Phi_0(x,\ y,\ z)}\right] = C\,\frac{\Phi_1(x,\ y,\ z)\quad \Psi_1(x,\ y,\ z)}{\Phi_0^2(x,\ y,\ z)} \tag{6}$$

where, as before, C must be a constant, we shall find that, according as we use the first set of zeros or the second set of zeros of $\Psi_0(x,\ y,\ z)$, which are those of $\psi_0(x,\ y,\ z)$ and $\psi_2(x,\ y,\ z)$,

$$C = \frac{\Phi_0(0,\ 0,\ 0)\,\dfrac{\partial}{\partial z}\Psi_0(0,\ 0,\ 0)}{\Phi_1(0,\ 0,\ 0)\quad \Psi_1(0,\ 0,\ 0)}$$

or

$$C = \frac{\Phi_1(0,\ 0,\ 0)\,\dfrac{\partial}{\partial z}\Psi_0(0,\ 0,\ 0)}{\Phi_0(0,\ 0,\ 0)\quad \Psi_1(0,\ 0,\ 0)}.$$

But these two are not the same, since

$$\Phi_0^2(0,\ 0,\ 0) \neq \Phi_1^2(0,\ 0,\ 0).$$

Hence we conclude that no relations of the type (6) exist.

Quadratic relations between Φ_0, Φ_1, Ψ_0, Ψ_1 also seem not to exist for the same reason, viz. because the periods of Φ_0 and Φ_1 are smaller than those of Ψ_0 and Ψ_1.

It may be mentioned in this connection, that the following symmetrical quartic relation between the functions Φ_0, Φ_1, Ψ_0, Ψ_1 was discovered in the course of the work, which, as far as could be tested, satisfied all the conditions imposed; viz.

$$\Phi_0^2\,\Phi_1^2 - \Psi_0^2\,\Psi_1^2 = A\left[\Phi_0^4 + \Phi_1^4 - \Psi_0^4 - \Psi_1^4\right]$$

when A is a constant whose value can be obtained readily.

IV.

Consider the holomorphic function

$$f_1(x,\ y,\ z) = \prod_{k=0}^{k=\infty}\left(1 + e^{2a(2k+1)^3 + 2\pi i\left[\frac{3x}{\omega_1}(2k+1)^2 + \frac{4y}{\omega_2}(2k+1) + \frac{4z}{\omega_3}\right]}\right)$$

where the real part of a is negative. It is obvious that

$$S_1 f_1(x,\ y,\ z) = \frac{1}{1 + e^{2a + 2\pi i\left(\frac{3x}{\omega_1} + \frac{4y}{\omega_2} + \frac{4z}{\omega_3}\right)}}\,f_1(x,\ y,\ z).$$

Again, writing

$$f_2(x, y, z) = \prod_k \left(1 + e^{2a(2k+1)^3 - 2\pi i \left[\frac{3x}{\omega_1}(2k+1)^2 - \frac{4y}{\omega_2}(2k+1) + \frac{4z}{\omega_3}\right]}\right)$$

we see at once that

$$S_1 f_2(x, y, z) = \left[1 + e^{-2a - 2\pi i \left(\frac{3x}{\omega_1} + \frac{4y}{\omega_2} + \frac{4z}{\omega_3}\right)}\right] f_2(x, y, z).$$

Finally, writing

$$F(x, y, z) = f_1(x, y, z) . f_2(x, y, z)$$

we have

$$S_1 F(x, y, z) = \frac{1 + e^{-2a - 2\pi i \left(\frac{3x}{\omega_1} + \frac{4y}{\omega_2} + \frac{4z}{\omega_3}\right)}}{1 + e^{2a + 2\pi i \left(\frac{3x}{\omega_1} + \frac{4y}{\omega_2} + \frac{4z}{\omega_3}\right)}} F(x, y, z)$$

or

$$S_1 F(x, y, z) = e^{-2a - 2\pi i \left(\frac{3x}{\omega_1} + \frac{4y}{\omega_2} + \frac{4z}{\omega_3}\right)} F(x, y, z).$$

We also have

$$F\left(x + \frac{\omega_1}{3}, y, z\right) = F\left(x, y + \frac{\omega_2}{4}, z\right) = F\left(x, y, z + \frac{\omega_3}{4}\right) = F(x, y, z).$$

If we put

$$2a = A, \quad \frac{\omega_1}{3} = \Omega_1, \quad \frac{\omega_2}{4} = \Omega_2, \quad \frac{\omega_3}{4} = \Omega_3$$

our function becomes

$$X(x, y, z) =$$
$$\prod_k \left[\left(1 + e^{A(2k+1)^3 + 2\pi i \left[\frac{x}{\Omega_1}(2k+1)^2 + \frac{y}{\Omega_2}(2k+1) + \frac{z}{\Omega_3}\right]}\right)\left(1 + e^{A(2k+1)^3 - 2\pi i \left[\frac{x}{\Omega_1}(2k+1)^2 - \frac{y}{\Omega_2}(2k+1) + \frac{z}{\Omega_3}\right]}\right)\right].$$

Now we have

$$X(x + \Omega_1, y, z) = X(x, y + \Omega_2, z) = X(x, y, z + \Omega_3) = X(x, y, z),$$

and, denoting by T_p the resulting form of S_p, viz.

$$T_p = \left(x, y, z, x + \frac{3A\Omega_1}{\pi i} p, y + \frac{4x\Omega_2}{\Omega_1} p + \frac{6A\Omega_2}{\pi i} p^2, z + \frac{2y\Omega_3}{\Omega_2} p + \frac{4x\Omega_3}{\Omega_1} p^2 + \frac{4A\Omega_3}{\pi i} p^3\right)$$

we have

$$T_1 X(x, y, z) = e^{-A - 2\pi i \left(\frac{x}{\Omega_1} + \frac{y}{\Omega_2} + \frac{z}{\Omega_3}\right.} X(x, y, z).$$

The function $X(x, y, z)$ which resembles the functions already considered, in being periodic, and in being reproduced to within a factor on being subjected to a linear substitution, seems to differ from them in not satisfying any simple differential equation or equations. There seems also to be more freedom in obtaining the zeros of this function. Thus, while in the case of the functions already considered, only y or z separately could be taken arbitrary, these two variables having only to satisfy one condition, and x had to be chosen subject to an independent condition, our present function $X(x, y, z)$ vanishes whenever either of the following conditions is satisfied:

$$A(2k+1)^3 + 2\pi i\left[\frac{x}{\Omega_1}(2k+1)^2 + \frac{y}{\Omega_2}(2k+1) + \frac{z}{\Omega_3}\right] = (2l+1)\pi i$$

or

$$A(2k+1)^3 - 2\pi i\left[\frac{x}{\Omega_1}(2k+1)^2 - \frac{y}{\Omega}(2k+1) + \frac{z}{\Omega_3}\right] = (2l+1)\pi i$$

where l is any integer, positive, zero or negative, and k is any positive integer, including zero. The analogy, however, between our functions is made the more striking by noticing that the zeros we found for $\varphi_0(x, y, z)$ are also such for $X(x, y, z)$. But all the zeros of the latter are not included in the former.

It may be interesting to note still further the similarities and the dissimilarities existing between our two classes of functions. For the sake of brevity write

$$E_1(k) = A(2k+1)^3 + 2\pi i\left[\frac{x}{\Omega_1}(2k+1)^2 + \frac{y}{\Omega_2}(2k+1) + \frac{z}{\Omega_3}\right]$$

$$E_2(k) = A(2k+1)^3 + 2\pi i\left[\frac{x}{\Omega_1}(2k+1)^2 + \frac{y}{\Omega_2}(2k+1) + \frac{z}{\Omega_3}\right].$$

Increasing z by $\frac{\Omega_3}{2}$ we get a new function

$$X(x, y, z+\frac{\Omega_3}{2}) \equiv X_1(x, y, z) = \prod_k\left[(1-e^{E_1(k)})(1-e^{E_2(k)})\right].$$

And as before

$$X(x+\frac{\Omega_1}{2}, y, z) = X(x, y+\frac{\Omega_2}{2}, z) = X(x, y, z+\frac{\Omega_3}{2})$$
$$= X(x+\frac{\Omega_1}{2}, y+\frac{\Omega_2}{2}, z+\frac{\Omega_3}{2}) = X_1(x, y, z),$$

$$X(x, y+\frac{\Omega_2}{2}, z+\frac{\Omega_3}{2}) = X(x+\frac{\Omega_2}{2}, y, z+\frac{\Omega_3}{2}) = X(x+\frac{\Omega_1}{2}, y+\frac{\Omega_2}{2}, z)$$
$$= X(x, y, z) \equiv X_0(x, y, z).$$

The effect of the substitution T_p is to change

$$E_1(k) \text{ into } E_1(k+p), \text{ and } E_2(k) \text{ into } E_2(k-p)$$

$$\therefore \quad T_p X_0(x, y, z) = \frac{[1+e^{-E_1(0)}][1+e^{-E_1(1)}]\ldots[1+e^{-E_1(p-1)}]}{[1+e^{E_1(0)}][1+e^{E_1(1)}]\ldots[1+e^{E_1(p-1)}]} X_0(x, y, z)$$
$$= e^{-[E_1(0)+E_1(1)+\ldots+E_1(p-1)]} X_0(x, y, z).$$

And, in the same way

$$T_p X_1(x, y, z) = (-1)^p \, e^{-[E_1(0) + E_1(1) + \dots + E_1(p-1)]} X_1(x, y, z).$$

The effect of the substitution

$$T_{\frac{1}{2}} = (x, y, z;\; x + \frac{3A\Omega_1}{2\pi i},\; y + \frac{2x\Omega_2}{\Omega_1} + \frac{3A\Omega_2}{2\pi i},\; z + \frac{y\Omega_3}{\Omega_2} + \frac{x\Omega_3}{\Omega_1} + \frac{A\Omega_3}{2\pi i})$$

is to change

$$E_1(k) \text{ into } E_1(k + \tfrac{1}{2}), \text{ and } E_2(k) \text{ into } E_2(k - 1)$$

thus giving rise, as before, to entirely new functions, when applied to $X_0(x, y, z)$ and $X_1(x, y, z)$:

$$T_{\frac{1}{2}} X_0(x, y, z) \equiv \Xi_0(x, y, z) = \prod_k [(1 + e^{E_1(k+\frac{1}{2})})(1 + e^{E_2(k-\frac{1}{2})})]$$

$$= \prod_k [(1 + e^{A(2k+2)^3 + 2\pi i [\frac{x}{\Omega_1}(2k+2)^2 + \frac{y}{\Omega_2}(2k+2) + \frac{z}{\Omega_3}]})(1 + e^{A(2k)^3 - 2\pi i [\frac{x}{\Omega_1}(2k)^2 - \frac{y}{\Omega_2}(2k) + \frac{z}{\Omega_3}]})]$$

and

$$T_{\frac{1}{2}} X_1(x, y, z) \equiv \Xi_1(x, y, z) = \prod_k [(1 - e^{E_1(k+\frac{1}{2})})(1 - e^{E_2(k-\frac{1}{2})})].$$

It is noticeable that instead of the periods of $\Xi(x, y, z)$ being multiples of those of $X(x, y, z)$ we have

$$\Xi(x + \frac{\Omega_1}{4}, y, z) = \Xi(x, y + \frac{\Omega_2}{2}, z) = \Xi(x, y, z + \Omega_3) = \Xi(x, y, z).$$

Again,

$$\Xi_0(x, y, z + \frac{\Omega_3}{2}) = \Xi_1(x, y, z),$$

$$\Xi_1(x, y, z + \frac{\Omega_3}{2}) = \Xi_0(x, y, z);$$

but there seems no way of passing from $\Xi_0(x, y, z)$ to $\Xi_1(x, y, z)$, or vice versa, by a change in x or y.

As in the case of $X(x, y, z)$, we have

$$T_p \Xi_0(x, y, z) = e^{-[E_1(\frac{1}{2}) + E_1(\frac{3}{2}) + \dots + E_1(p-\frac{1}{2})]} \Xi_0(x, y, z)$$

$$T_p \Xi_1(x, y, z) = (-1)^p e^{-[E_1(\frac{1}{2}) + E_1(\frac{3}{2}) + \dots + E_1(p-\frac{1}{2})]} \Xi_1(x, y, z).$$

Also

$$T_{\frac{1}{2}} \Xi_0(x, y, z) = \prod_k [(1 + e^{E_1(k+1)})(1 + e^{E_2(k-1)})]$$

$$= e^{-E_1(0)} X_0(x, y, z) = T_1 X_0(x, y, z)$$

$$T_{\frac{1}{2}} \Xi_1(x, y, z) = -e^{-E_1(0)} X_1(x, y, z) = T_1 X_1(x, y, z).$$

But we had, by definition

$$T_{\frac{1}{2}} X_0(x, y, z) = \Xi_0(x, y, z), \quad T_{\frac{1}{2}} X_1(x, y, z) = \Xi_1(x, y, z).$$

Hence we see that

$$T_{\frac{1}{2}}^2 X(x, y, z) = T_1 X(x, y, z)$$

and similarly

$$T_{\frac{1}{2}}^2 \Xi(x, y, z) = T_1 \Xi(x, y, z)$$

i. e. the effect of two successive operations of $T_{\frac{1}{2}}$ is identical with that of a single application of T_1.

Changing the sign of x and z simultaneously interchanges E_1 and E_2. From this follows that X_0 and X_1 are even as to x and z simultaneously. But for $\Xi(x, y, z)$, we have the values changed, thus

$$\Xi_0(-x, y, -z) = \frac{1 + e^{E_1(-\frac{1}{2})}}{1 + e^{E_2(-\frac{1}{2})}} \Xi_0(x, y, z)$$

$$\Xi_1(-x, y, -z) = \frac{1 - e^{E_1(-\frac{1}{2})}}{1 - e^{E_2(-\frac{1}{2})}} \Xi_1(x, y, z).$$

In conclusion, it may be mentioned that by taking the logarithmic derivative of $X_0(x, y, z)$ with respect to z, we shall obtain a new function analogous to the Z-function of one variable. Thus, writing

$$X_0(x, y, z) = \Pi \left\{ 1 + e^{2A(2k+1)^3 + 4\pi i \frac{y}{\Omega_2}(2k+1)} + 2e^{A(2k+1)^3 + 2\pi i \frac{y}{\Omega_2}(2k+1)} \cos\left[2\pi \frac{x}{\Omega_1}(2k+1)^2 + \frac{z}{\Omega_3}\right] \right\}$$

we have

$$\frac{\frac{\partial}{\partial z} X_0(x, y, z)}{X_0(x, y, z)} \equiv \Lambda(x, y, z) =$$

$$-\frac{z}{\Omega_3} \sum_{k=0}^{k=\infty} \frac{e^{A(2k+1)^3 + 2\pi i \frac{y}{\Omega_2}(2k+1)} \sin\left[2\pi \frac{x}{\Omega_1}(2k+1)^2 + \frac{z}{\Omega_3}\right]}{1 + e^{2A(2k+1)^3 + 4\pi i \frac{y}{\Omega_2}(2k+1)} + 2e^{A(2k+1)^3 + 2\pi i \frac{y}{\Omega_2}(2k+1)} \cos\left[2\pi \frac{x}{\Omega_1}(2k+1)^2 + \frac{z}{\Omega_3}\right]}.$$

This series is uniformly convergent, since the real part of A is negative, and therefore represents a function. We have, evidently,

$$\Lambda(x + \Omega_1, y, z) = \Lambda(x, y + \Omega_2, z) = \Lambda(x, y, z + \Omega_3) = \Lambda(x, y, z)$$

and

$$T_1 \Lambda(x, y, z) = \frac{\frac{\partial}{\partial z}\left[e^{-\Lambda - 2\pi i\left(\frac{x}{\Omega_1} + \frac{y}{\Omega_2} + \frac{z}{\Omega_3}\right)}\right]}{e^{-\Lambda - 2\pi i\left(\frac{x}{\Omega_1} + \frac{y}{\Omega_2} + \frac{z}{\Omega_3}\right)}} + \Lambda(x, y, z) = -\frac{2\pi i}{\Omega_3} + \Lambda(x, y, z).$$

Another differentiation will give us a doubly periodic function, for

$$\frac{\partial}{\partial z}\Lambda(x, y, z) = \frac{\partial}{\partial z}\Lambda(x + \Omega_1, y, z) = \frac{\partial}{\partial z}\Lambda(x, y + \Omega_2, z) = \frac{\partial}{\partial z}\Lambda(x, y, z + \Omega_3)$$

and

$$T_1 \frac{\partial}{\partial z}\Lambda(x, y, z) = \frac{\partial}{\partial z}\Lambda(x, y, z).$$

The successive derivatives also have the same property.

Biographical Sketch.

The author, Abraham Cohen, was born in Baltimore, Md., September 11, 1870. His elementary education was gotten at Scheib's Zion School, where he was enrolled from 1877 till 1883. He then entered the Baltimore City College, and upon graduation, in 1888, was admitted as a candidate for the degree of Bachelor of Arts, in the Johns Hopkins University. This degree was conferred upon him in June, 1891, and in the fall of that year he re-entered this University, as a candidate for the degree of Doctor of Philosophy, selecting Mathematics as his principal, and Astronomy and Physics as his first and second subordinate subjects respectively. Upon receiving his Bachelor's degree he was awarded a University scholarship, which he resigned to accept an appointment as Assistant in Mathematics. This position he held for two years. During the past year he has held the Fellowship in Mathematics.