A Stronger Limit Theorem for Stable Population Theory

by

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Abstract

The discrete model $x_{t+1} = Ax_t$ for the dynamics of the age composition of populations with constant age-specific vital rates over time leads to the limiting result

$$\lim_{t\to\infty} x_t/r^t = u(v,x_o) \ .$$

Necessary and sufficient conditions for the validity of

$$\lim_{t\to\infty} [x_t - r^t u(v,x_o)] = 0 \ ,$$

which is stronger for populations with $r > 0$, are given in Theorem 4; the sufficient condition $|\lambda_j|<1$ for $\lambda_j \neq r$ holds empirically for human populations, and further justifies use of the approximation.

$$x_t \doteq r^t u(v,x_o)$$

in demographic work. Consideration of the two limits makes it clear that the behavior of x_t should be discussed separately for the cases $r \lesseqqgtr 1$.

Because of the preference of many demographers and biologists for the continuous version of stable population theory, analogous results for the birth density b(t) are presented in Theorem 7.

§1. Introduction

The approximation

$$x_t \doteq r^t u(v, x_o) \quad , \tag{1}$$

routinely used in demography, expresses age distribution x_t at time t in terms of an initial age distribution x_o and the positive eigenvalue r and corresponding positive eigenvectors u and v of a population projection matrix. Justification for the use of this approximation is provided by a theorem concerning the asymptotic behavior of the sequence $\{x_t/r^t\}$ of normalized vectors, which asserts that the relative error of the approximation becomes arbitrarily small as t increases. For populations with $r > 1$, a stronger limit theorem given in §3 asserts that when its conditions are satisfied, the absolute error of the approximation tends to zero as t increases. The theorem is trivial mathematically, and depends for its significance on the empirical observation that its sufficient condition holds for every population projection matrix contained in the compendia of Keyfitz and Flieger ([1968], [1971]).

To discuss the dynamics of a closed population with constant age-specific birth and death rates in terms of the discrete formulation

$$x_{t+1} = Ax_t \quad , \qquad t = 0,1,\ldots \tag{2}$$

is largely a matter of taste; an entirely equivalent discussion would arise from consideration of the continuous formulation

$$b(t) = g(t) + \int_0^t \varphi(s)b(t-s)ds \ , \qquad t \geq 0 \tag{3}$$

for the birth density b(t) in terms of the functions g(t) and φ(t), with the approximation

$$b(t) \doteq ae^{rt} \quad . \tag{4}$$

Because of the popularity of this latter formulation of the problem among both demographers and biologists, sufficient conditions for the continuous analogues of discrete results to hold are given in the final section of this paper.

§2. Preliminaries for the discrete case

In equation (2), A is a population projection matrix, defined in terms of its elements a_{ij} by

$$a_{ij} = \begin{cases} b_j \ , \ i = 1, \ j = 1,\dots,n \\ s_j \ , \ i = 2,\dots,n, \ j = i-1 \\ 0 \ , \ \text{otherwise} \end{cases}$$

where $b_j \geq 0$, $b_n > 0$ and $0 < s_j \leq 1$. Properties of such matrices have been discussed in detail recently by a number of authors, among them Goodman [1967], Keyfitz [1968], Pollard [1973], and Sykes [1969]. Although the simple structure of population projection matrices makes the restriction to finite dimensional spaces unnecessary mathematically, it appears eminently reasonable biologically, and the discrete model (2) has enjoyed considerable popularity in the literature. See Feller ([1968], Chapter XIII) for the extension of the theory to the denumerable case.

Right and left eigenvectors of A will be denoted respectively by u_j and v_j, and the corresponding eigenvalues of A by λ_j; i.e., $Au_j = \lambda_j u_j$ and $v_j A = \lambda_j v_j$, where λ_j is a solution of the characteristic equation

$$\sum_{i=1}^{n} p_i b_i \lambda^{-i} = 1 \tag{5}$$

with

$$p_i = \begin{cases} 1 \ , & i = 1 \\ \prod_{k=1}^{i-1} s_k \ , & i = 2,\dots,n \end{cases} .$$

Equation (5) has exactly n roots, none of which is zero because $p_n b_n > 0$, and hence the eigenvectors u_j and v_j exist. In general, the roots

$\lambda_1, \ldots, \lambda_n$ are complex numbers; if a given λ_j is not real, then its complex conjugate $\bar{\lambda}_j$ is also a root. From the eigenvector equations, it follows that if u_j and v_j correspond to λ_j, then $\bar{u}_j$ and $\bar{v}_j$ are the eigenvectors corresponding to $\bar{\lambda}_j$. Although it is not necessary for the discussion of the limiting behavior of the sequences $\{x_t\}$and $\{x_t/r^t\}$, it is convenient to keep in mind the expressions given by Goodman [1967] for the elements of the eigenvectors, in which the first element of each right eigenvector u_j is taken as unity, and left eigenvectors are normalized to give $\sum_i u_{ji} v_{ji} = 1$; these expressions are

$$u_{ji} = p_i \lambda_j^{-i+1}$$

and

$$v_{ji} = \sum_{k=i}^{n} p_k b_k \lambda_j^{-k} / \mu_j u_{ji} ,$$

where

$$\mu_j = \sum_{i=1}^{n} \sum_{k=i}^{n} p_k b_k \lambda_j^{-k}$$
$$= \sum_{i=1}^{n} i p_i b_i \lambda_j^{-i} .$$

By convention, $r = \lambda_1$ denotes the positive root of equation (5), and the eigenvectors corresponding to r are denoted by unsubscripted u and v when they appear alone.

In general, vectors will be considered as elements of $\mathbb{C}^n$, an n-dimensional vector space over the field of complex numbers. The notation $x > 0$ means $x \neq 0$ and $x_i \geq 0$, while $x \gg 0$ implies that all elements of x are positive. For the inner product of two elements x and y (both considered as column-vectors) of the space, the standard notation (x,y) will be used, so that

$$(x,y) = \sum_k x_k \bar{y}_k .$$

(This notation has been chosen to make more obvious the analogy between solutions in the discrete and continuous cases; readers unfamiliar with it may read "(x,y)" as "$x\bar{y}$," where x is taken as a row- and y as a column-vector.) Repeated use will be made of the readily verified properties of inner products that

$$(x,y) = \overline{(y,x)} = (\bar{y},\bar{x}) \quad ,$$

that

$$(kx,y) = k(x,y)$$

for scalar k, and that

$$(x,My) = (M^{*}x,y)$$

for a matrix M, where M^{*} denotes the conjugate transpose of M. Note that the first two properties imply that

$$(x,ky) = \bar{k}(x,y)$$

and that if $xM = \lambda x$ for real M, the second and third imply that

$$(x,My) = \lambda\,(x,y).$$

Two vectors are said to be orthogonal (notation: $x \perp y$) if and only if $(x,y) = 0$; similarly x is said to be orthogonal to a subspace S of $\mathbb{C}^n$ if and only if $(x,y) = 0$ for every y in S. In particular, a right eigenvector u_i, corresponding to the eigenvalue λ_i of a real matrix M, is orthogonal to all left eigenvectors of M except $\bar{v}_i$, the one corresponding to $\bar{\lambda}_i$. Since it is always possible to normalize so that $(u_i,\bar{v}_i) = 1$, in general

$$(u_i,\ \bar{v}_j) = \delta_{ij} \quad ,$$

where $\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ij} = 1$ when $i = j$. It can easily be verified that the expressions given for u_j and v_j above satisfy $(u_i,\ \bar{v}_j) = \delta_{ij}$, and hence these vectors $u_1,\ldots,u_n$ and $v_1,\ldots,v_n$ form a normalized biorthogonal system for $\mathbb{C}^n$ whenever the eigenvalues of A are distinct. Note also that for real x, if $(x,v_j) = 0$ then $(x,\bar{v}_j) = 0$, so that if $x \perp v_j$ then $x \perp S_j$, the two-dimensional subspace of $\mathbb{C}^n$ spanned by

v_j and $\bar{v}_j$, and hence x can be written as a linear combination of right eigenvectors u_i excluding u_j and $\bar{u}_j$.

§3. Limit theorems for the discrete case

For a given value of x_o, the solution of equation (2) is

$$x_t = A^t x_o \quad , \qquad t = 0,1,\ldots \tag{6}$$

so that the behavior of the sequence $\{x_t\}$ can be, and usually is, discussed in terms of that of the sequence $\{A^t\}$. The standard result is

THEOREM 1. Let A be a population projection matrix with positive eigenvalue r and positive (right and left) eigenvectors $u \gg 0$ and $v \gg 0$. Then

$$\lim_{t\to\infty} A^t/r^t = uv$$

if and only if A is aperiodic.

Because a sufficient condition for aperiodicity (also called "primitivity") is that fertility be positive for two adjacent age groups (i.e., that both $b_j > 0$ and $b_{j+1} > 0$ for some $j = 1,\ldots,n-1$), Theorem 1 applies to matrices A for human populations in practice. When A is periodic with period $d \leq n$, the results are (Cox and Miller [1965] p. 123)

$$\lim_{t\to\infty} A^{td+s}/r^{td+s} = \sum_{j=1}^{d} e^{2\pi i s(j-1)/d} u_j v_j \ , \quad s = 0,\ldots,d-1$$

(in particular, for s = 0, $\lim_{t\to\infty} A^{td}/r^{td} = \sum_{j=1}^{d} u_j v_j$) and

$$\lim_{t\to\infty} \frac{1}{d} \sum_{s=0}^{d-1} A^{td+s}/r^{td+s} = uv$$

where eigenvectors corresponding to eigenvalues with $|\lambda_j| = r$ have been numbered $1,\ldots,d$. Because discussion here is primarily of eigenvalues satisfying $|\lambda_j|<r$, strongest results for periodic matrices A are not included.

Theorem 1 implies, but is not necessary for

THEOREM 2. Let A be an aperiodic population projection matrix with positive eigenvalue r and positive eigenvectors u and v, and let $x_t = A^t x_o$, where $x_o > 0$. Then

$$\lim_{t\to\infty} x_t/r^t = u(v,x_o) \quad .$$

Theorem 2 is taken as justification for the use of the approximation

$$x_t \doteq r^t u(v,x_o) \tag{1}$$

given in §1.

A closer look at Theorem 2 will clarify the sense in which the approximation (1) is a good one. Write

$$e_t = x_t - r^t u(v,x_o) \quad . \tag{7}$$

Then by Theorem 2 the relative error in the i-th element of e_t, $e_{ti}/r^t u_i(v,x_o)$, satisfies

$$\lim_{t\to\infty} \frac{e_{ti}}{r^t u_i(v,x_o)} = \lim_{t\to\infty} \frac{x_{ti}}{r^t u_i(v,x_o)} - 1$$

$$= 0;$$

i.e., given A and x_o, for each $\epsilon > 0$ we can find a value T such that $|e_{ti}/r^t u_i(v,x_o)|<\epsilon$ for all $t > T$. In other words, the approximation (1)

To simplify discussion of $\{e_t\}$, assume further that the eigenvalues of A are distinct, so that the eigenvectors u_j are linearly independent (Gantmacher [1959], p. 72). Then an arbitrary vector x_t, considered as an element of $\mathbb{C}^n$, can be written

$$x_t = \sum_{j=1}^{n} a_{tj} u_j \qquad ,$$

where the coefficients a_{tj} of x_t relative to the basis $u_1, \ldots, u_n$ are found by noting that

$$\begin{aligned}(v_j, x_t) &= (x_t, \overline{v}_j) \\ &= (\sum_i a_{ti} u_i, \overline{v}_j) \\ &= a_{tj}\end{aligned}$$

since $(u_i, \overline{v}_j) = \delta_{ij}$. On the other hand,

$$\begin{aligned}(v_j, x_t) &= (v_j, A^t x_o) \\ &= \lambda_j^t (v_j, x_o)\end{aligned}$$

since A is real, so that

$$x_t = \sum_{j=1}^{n} \lambda_j^{\,t} (v_j, x_o) u_j \qquad . \tag{8}$$

(Equation (8) follows equivalently, and more usually in the literature, from the similarity of A to a diagonal matrix if its eigenvalues are distinct.) From equations (7) and (8),

$$e_t = \sum_{j=2}^{n} \lambda_j^{\,t} (v_j, x_o) u_j \tag{9}$$

(Note that e_t is thus an element of an (n-1)-dimensional subspace of $\mathbb{C}^n$, which is spanned by $u_2, \ldots, u_n$.)

It is clear from equation (9) that $\{e_t\}$ may diverge if $|\lambda_j| \geq 1$ for

$\lambda_j \neq r$, but that it will not if $(v_j,x_o) = 0$ for each eigenvector corresponding to an eigenvalue different from r for which $|\lambda_j| \geq 1$. Because r is the only positive eigenvalue of A, such $\lambda_j \neq 1$, and so it also follows that if $\{e_t\}$ converges, it converges to the zero vector. These considerations provide the motivation for

THEOREM 3. Let A be an aperiodic population projection matrix with distinct eigenvalues, and let $x_t = A^t x_o$, where $x_o > 0$. Then $\{e_t\}$ defined by equation (7) converges if and only if, for each $\lambda_j \neq r$, either $|\lambda_j| < 1$ or $(v_j,x_o) = 0$; under this condition,

$$\lim_{t\to\infty} [x_t - r^t u(v,x_o)] = 0.$$

<u>Proof</u>: If, for $j = 2,\ldots,n$, either $|\lambda_j|<1$ or $(v_j,x_o) = 0$, then $\lim \lambda_j^t(v_j,x_o) = 0$, and so sufficiency is established. For the necessity of the condition, suppose that $\{e_t\}$ converges to some vector e in the subspace spanned by $u_2,\ldots,u_n$. Because $u_2,\ldots,u_n$ are linearly independent, the coefficients a_j of e with respect to them are uniquely determined, and hence $\lim \lambda_j^t(v_j,x_o) = a_j$ exists for each of $j = 2,\ldots,n$. But $\lim \lambda_j^t(v_j,x_o)$ exists only if either $|\lambda_j|<1$ (since $\lambda_j \neq 1$) or $(v_j,x_o) = 0$, and in either case $a_j = 0$, which implies that $e = 0$.

Modifications necessary in Theorem 3 to include periodic matrices are minor.

Although Theorem 3 was stated without reference to the value of r, it is trivial when $r \leq 1$, since by the Perron-Frobenius theorem the sufficient condition $|\lambda_j|<1$ holds then. Three cases may thus be distinguished, according as to whether $r \lesseqgtr 1$; when $r \leq 1$, the requirement of distinct eigenvalues in Theorem 3 is unnecessary. Furthermore, because

$$x_t - r^t u(v,x_o) = r^t [\frac{x_t}{r^t} - u(v,x_o)] \quad , \tag{10}$$

which of Theorems 2 and 3 is the stronger depends also on the value of r; if $r \leq 1$, Theorem 2 implies Theorem 3, while if $r \geq 1$ the reverse implication is true.

Note that when $r > 1$, the well-known results concerning proportional age structure $c_t = x_t/\sum_k x_{tk}$, birth and death rates b_t and d_t, and the one-step rate of natural increase $\rho_t = b_t - d_t$ follow from the weaker Theorem 2. Indeed, for the population in the n reproductive age groups,

$$\begin{aligned} c &= \lim_{t\to\infty} c_t \\ &= \lim_{t\to\infty} \frac{x_t/r^t}{\sum_k x_{tk}/r^t} \\ &= u/\sum_k u_k \quad ; \end{aligned}$$

similarly,

$$\begin{aligned} b &= \lim_{t\to\infty} b_t \\ &= \lim_{t\to\infty} \frac{\Sigma b_k x_{tk}}{\Sigma x_{tk}} \\ &= r/\Sigma u_k \end{aligned}$$

and, taking $s_n = 0$,

$$\begin{aligned} d &= \lim_{t\to\infty} d_t \\ &= \lim_{t\to\infty} \frac{\Sigma(1-s_k)x_{tk}}{\Sigma x_{tk}} \\ &= 1 - r + r/\Sigma u_k \quad ; \end{aligned}$$

finally

$$\begin{aligned} \rho &= \lim_{t\to\infty} \rho_t \\ &= \lim_{t\to\infty} (b_t - d_t) \\ &= r - 1 \quad , \end{aligned}$$

each independently of initial age structure x_o. On the other hand, convergence of the age structure x_t, and in particular of the "birth" sequence $\{x_{t1}\}$, to a sequence which depends on initial age structure only through the scalar (v,x_o) requires Theorem 3 when $r > 1$. Similarly, when $r < 1$ the above results require the stronger Theorem 2, while convergence of the age structure (to the zero vector!) follows from Theorem 3.

Case i: $r < 1$. From the Jordan form (Gantmacher [1959], pp. 151-3) for A, or from the fact that r is the spectral radius (Karlin [1966], pp. 479-80) of A, it follows that $\lim_{t\to\infty} x_t = 0$, and of course $\lim_{t\to\infty} r^t u(v,x_o) = 0$, so

$$\lim_{t\to\infty} [x_t - r^t u(v,x_o)] = 0$$

follows trivially from a theorem on the limit of sums. The conclusion of Theorem 3 holds for any population projection matrix with $r < 1$, and although Theorem 2 also applies, note that by equation (10) the absolute error of the approximation (1) is less than the relative error when $r < 1$. It is also true in this case that $\sum_t x_{t1}$ converges (Feller [1968], p. 330), and so the strongest result when $r < 1$ is

$$\sum_{t=1}^{\infty} x_{t1} = \frac{(v,x_o)}{1 - N}$$

where $N = \Sigma p_i b_i$ is the net reproduction rate implied by A.

Case ii: $r = 1$. Theorems 2 and 3 are equivalent when $r = 1$, both give

$$\lim_{t\to\infty} x_t = u(v,x_o) \quad .$$

Stronger results may be found in Feller ([1968], Chapter XIII).

Case iii: $r > 1$. By equation (10), when $r > 1$ the relative error of approximation (1) is less than the absolute error, and so Theorem 3 is stronger than Theorem 2. Only in this case do the eigenvalues of A other than r play an important role in determining the behavior of $\{e_t\}$.

Three general types of behavior for $\{e_t\}$, and hence of $\{x_t\}$, are thus possible. They are illustrated by Figures 1-3, which show values of $\{e_{t1}\}$, the errors in the discrete "birth" sequence, for 100 five-year steps (point values have been connected by straight lines even though calculations were made only at integer values of t) using the net maternity function for Japan, 1964, adjusted to give, respectively, $\max_{\lambda_j \neq r} |\lambda_j| < 1$, $= 1$, and > 1 (details of the calculations may be found in the Appendix). It is perhaps unnecessary to point out that Theorem 2 guarantees that $\{e_t/r^t\}$ always behaves like $\{e_t\}$ in Figure 1. If $r \leq 1$, $\{e_t\}$ converges to zero in the damped oscillatory manner of Figure 1, as is also the case when $r > 1$ but $|\lambda_j| < 1$ or $(v_j, x_o) = 0$ for $\lambda_j \neq r$. Then also $\{x_t\}$ converges to $\{r^t u(v, x_o)\}$. If $r > 1$ and $|\lambda_j| = 1$ for one or more $\lambda_j \neq r$, then $\{e_t\}$ remains bounded and eventually oscillates around zero; for example, if only one of $\lambda_2, \ldots \lambda_n$ satisfies $|\lambda_j| = 1$, then for any norm,

$$\lim_{t \to \infty} \|e_t\| = |(v_j, x_o)| \cdot \|u_j\|$$

as illustrated by Figure 2. It follows that in this case $\{x_t\}$ will oscillate around $\{r^t u(v, x_o)\}$. Finally, when $r > 1$ and $|\lambda_j| > 1$ for one or more $\lambda_j \neq r$, then $\{e_t\}$ diverges without bound, and hence $\{x_t\}$ will depart farther and farther from $\{r^t u(v, x_o)\}$ as t increases (see Figure 3).

The curious fact that matrices A corresponding to human populations satisfy the sufficient condition $|\lambda_j| < 1$, $\lambda_j \neq r$, of Theorem 3 was apparently first noted by Lotka ([1939], p. 66), although it is difficult to imagine that the number of cases available for his inspection was large. Somewhat stronger evidence is provided by the data given by Keyfitz and Flieger ([1968], [1971]) for a large number of human populations.

Calculation of all eigenvalues of each of the matrices given there (and of several others) produced none, other than the positive eigenvalue r, which was as large as unity in absolute value. It thus appears reasonably safe to assume that the conclusion of Theorem 3 applies to human populations, so that $\{x_t\}$ converges to $\{r^t u(v,x_o)\}$.

It is perhaps worth noting that, although the sufficient condition on the eigenvalues of A holds for human populations, the condition that x_o be orthogonal to one or more of the left eigenvectors $v_2,\ldots,v_n$ is also easy to describe. Recall that if λ_k and λ_m are complex conjugates, then so are v_k and v_m, and so any vector $x_o>0$ which can be written $x_o = \sum_{j\neq k,m} a_j u_j$ is orthogonal to the subspace spanned by v_k and v_m. It is easily verified that such vectors may be positive; an important example is the one-dimensional subspace of vectors x which satisfy x = ku, where $k > 0$, for which $x_t = r^t ku$ for all finite values of t. In practical questions regarding convergence of $\{x_t\}$ to $\{r^t u(v,x_o)\}$, it appears to be the case frequently that x_o is "almost orthogonal" to one or more left eigenvectors, so that $\{x_t\}$ converges quite rapidly regardless of the proximity of $\max_{\lambda_j\neq r}|\lambda_j|$ to the unit circle.

§4. Multiple eigenvalues

The possibility that the characteristic equation (5) may have roots of algebraic multiplicity greater than one seems largely a nuisance in population mathematics. Keyfitz ([1968], p. 51) reports that such cases do not arise among human populations in practice, and this is confirmed by an examination of the eigenvalues of the matrices given by Keyfitz and Flieger ([1968], [1971]). While the mathematics necessary to include matrices with multiple eigenvalues in the theorems of the preceding section is not difficult, it is rather technical, in that Jordan forms replace diagonal matrices and Jordan chains are used to generate the subspaces corresponding to eigenvalues of algebraic multiplicity greater than one, but geometric multiplicity one. It is also true that the specialized structure of population projection matrices allows a more direct proof of theorems like Theorem 4 than one which, like that of Theorems 1 to 3, treats the behavior of $\{x_t\}$ as a special case of the theory of irreducible non-negative matrices. For these reasons, although Theorem 4 extends the result of Theorem 3 to the case of matrices with multiple eigenvalues, proof is given only of the sufficiency of the condition $|\lambda_j|<1$, $\lambda_j \neq r$. A full proof along the lines of that of Theorem 3, and valid for irreducible non-negative matrices, is available elsewhere (Sykes [1975]). The statement of Theorem 4 incorporates both the adjustments necessary for multiple eigenvalues and the points made in the discussion following Theorem 3.

THEOREM 4. Let A be an aperiodic population projection matrix, with positive eigenvalue r and positive (right and left) eigenvectors u and v, and let $x_t = A^t x_o$, where $x_o > 0$. Then

(i) if $r < 1$, x_t is bounded for $t \geq 0$, and

(a) $\lim_{t\to\infty} x_t = 0$

(b) $\lim_{t\to\infty} x_t/r^t = u(v,x_o)$

(c) $\sum_{t=1}^{\infty} x_{t1} = \dfrac{(v,x_o)}{1 - N}$

(ii) if $r = 1$, x_t is bounded for $t \geq 0$, and

$$\lim_{t\to\infty} x_t = u(v,x_o)$$

(iii) if $r > 1$, x_t is bounded on any finite set $t = 0,1,\ldots,T$, while x_t/r^t is bounded for $t \geq 0$,

$$\lim_{t\to\infty} x_t/r^t = u(v,x_o)$$

and in addition $\{x_t - r^t u(v,x_o)\}$ converges and

$$\lim_{t\to\infty} [x_t - r^t u(v,x_o)] = 0$$

if and only if for $\lambda_j \neq r$ either $|\lambda_j|<1$ or x_o is orthogonal to the subspace S_j spanned by the left Jordan chain for A corresponding to λ_j.

<u>Proof</u>: Proof of all assertions except the last may be found in the preceding section; the sufficiency of the condition $|\lambda_j|<1$ for the last assertion is easily shown by considering the matrix $B = A - ruv$. A simple induction leads to $B^t = A^t - r^t uv$, so that equation (7) defining e_t may be written $e_t = B^t x_o$. Note that $Bx = 0$ implies $B^t x = 0$ if and only if $x = 0$ or $x = ku$, and that the eigenvalues of B are those of A which satisfy $\lambda_j \neq r$. Hence if $|\lambda_j|<1$, the spectral radius of B is less than unity, and so $\lim e_t = \lim B^t x_o = 0$.

Theorem 4 of course includes Theorem 3 as a special case, for when the eigenvalues of A are distinct, the subspace S_j is simply the one-dimensional space spanned by v_j. As in the simpler case, it is easy to characterize those vectors x_o which are orthogonal to a given subspace S_j; they are given by $x_o = \sum_{u_k \perp S_j} a_k u_k$, i.e., as a linear combination of those u_k which do not correspond to the eigenvalue λ_j. Note again that the existence of multiple eigenvalues is irrelevant when $r \leq 1$. Modifications of Theorem 4 to include periodic matrices should be obvious.

The behavior of $\{e_t\}$ determined by Theorem 4 is identical to that given after Theorem 3 with one exception: if $\max_{\lambda_j \neq r} |\lambda_j| = 1$, then $\{e_t\}$ diverges but is bounded if the multiplicity of λ_j is one, and diverges without bound if the multiplicity of λ_j is greater than one.

§5. Limit theorems for continuous stable population theory

In the continuous version of stable population theory (see, for example, Coale [1972] or Keyfitz [1968]), the birth density b(t) satisfies the integral equation

$$b(t) = g(t) + \int_0^t \varphi(s)b(t-s)ds \quad , \quad t \geq 0. \tag{11}$$

This equation is known (Riesz and Sz.-Nagy [1955], p. 147) to have a unique solution, given by a uniformly convergent Neumann series, on any finite interval on which g and φ are bounded. In the theory of probability, the equation plays a prominent role in renewal theory, where it is possible to extend the domain on which the solution is bounded to the non-negative real line because φ is taken to be a probability density. Although extensive accounts of renewal theory are available (Feller [1968], [1971], Smith [1958]), it has remained customary in the demographic literature to assume the existence of a solution to equation (11) of the form proposed by Lotka:

$$b(t) = \sum_{i=0}^{\infty} a_i e^{r_i t} \tag{12}$$

where r_i is a root of the characteristic equation

$$\int_0^\infty e^{-rs}\varphi(s)ds = 1 \tag{13}$$

and the numbers a_i are constants; necessary and sufficient conditions for the validity of equation (12) were given by Feller [1941], and discussed in the special context of demography by Lopez [1961]. Although the assumption that a solution of this form exists is quite restrictive mathematically, Lopez argued that its requirements were met in practical demography (even though it is not in fact used in numerical work, and its use in theoretical work has frequently been invalid); the solution has the additional advantage of providing an obvious analogue to equation (8) in the discrete case.

In equation (11), g is the contribution to b made by the initial population, and φ is the net maternity function

$$\varphi(a) = p(a)m(a) \quad ,$$

where $p(a)$ is the proportion of births which survive to age a and $m(a)$ is an appropriately defined fertility rate at a; it is usually assumed that p is continuous, monotonic and non-increasing, satisfying $1 \geq p(a) \geq 0$, and that m is continuous, non-negative and bounded, satisfying $0 \leq m(a) \leq M$. Recalling that x_{t1} is the discrete analogue of $b(t)$, and that normalized $u_{j1} = 1$, the analogy of the Lotka solution (12) and the discrete solution (8) can be completed by following Goodman [1967] in defining continuous functions u_i and v_i, for $i = 0,1,\ldots,$ by

$$u_i(a) = e^{-r_i a} p(a)$$

and

$$v_i(a) = \int_a^\infty e^{-r_i s} \varphi(s)ds/\mu_i u_i(a)$$

where

$$\mu_i = \int_o^\infty ae^{-r_i a} \varphi(a)da \quad .$$

The existence of these functions for values of a satisfying $0 \leq a \leq \omega$ is guaranteed if $p(a) = 0$ for $a > \omega$, as is usually assumed in demography;

note that this assumption implies that $\varphi(a) = 0$ for $a > \omega$. As Feller [1941] pointed out, the last assumption is sufficient for removing all the essential difficulties with the Lotka solution.

By convention, $r = r_o$ is the largest real root of equation (12), and its only real root if $\varphi(a)$ is positive only on a finite interval; subscripts will also be dropped from a_o, u_o and v_o when they appear alone. Defining the inner product of two complex-valued functions x and y of a real variable by

$$(x,y) = \int_o^\infty x(s)\bar{y}(s)ds \quad ,$$

it can be readily verified that the functions $u_o, u_1, \ldots$ and $v_o, v_1, \ldots$ satisfy

$$(u_i, \bar{v}_j) = \delta_{ij} \quad ,$$

so that these functions form a normalized biorthogonal system. Following the same arguments as in the discrete case, if a solution of the Lotka form (12) exists, then

$$a_i = (v_i, x(o))$$

where $x(o) = x(o,a)$ is the integrable initial age density. Thus (see also Coale [1972], pp. 67-8)

$$b(t) = \sum_{i=0}^{\infty} e^{r_i t}(v_i, x(o))$$

in exact analogy with the value of x_{t1} given by equation (8). By noting that the age density $x(t)$ at time t satisfies $x(t,a) = p(a)b(t-a)$ for $t \geq \omega$, this analogy can be extended to give

$$x(t) = \sum_{i=0}^{\infty} e^{r_i t} u_i(v_i, x(o)), \quad t \geq \omega \quad .$$

Except for questions of existence, which do not arise when φ is appropriately restricted, the limiting behavior of the Lotka solution thus appears to pose no new problems. However, instead of restricting φ to a finite interval, the more general apparatus of renewal theory, in which φ (and hence p) can have infinite domain, will be used in the following discussion of the behavior of $b(t)$ as $t \to \infty$.

The continuous analogue of Theorem 2 is

THEOREM 5. Let b(t) given by equation (12) be a solution of the renewal equation (11). Then

$$\lim_{t\to\infty} e^{-rt}b(t) = a \quad .$$

Theorem 5 is known to be true when equation (12) is valid, and it is in fact true much more generally whenever $r \geq 0$, and under conditions when $r < 0$ (for details, see Feller [1971], Chapter XI). The theorem seems to have been taken as the usual justification for the use of the approximation

$$b(t) \doteq e^{rt}(v,x(o))$$

in demography (Coale [1972], Chapter 3), although there is always a suggestion in the literature that demographers are aware of the validity in practice of the analogue of Theorem 3:

THEOREM 6. Let b(t) given by equation (12) be a solution of the renewal equation (11). Then if for each i either $a_i = 0$ or $\mathrm{Re}(r_i) < 0$,

$$\lim_{t\to\infty}[b(t) - ae^{rt}] = 0$$

(here $\mathrm{Re}(r_i)$ denotes the real part of r_i).

Note that Theorem 6 implies Theorem 5 if $r \geq 0$, while the reverse implication is true if $r \leq 0$.

As with the discrete formulation, stronger results can be obtained if separate consideration is given to the behavior of solutions according as $r \lesseqgtr 0$, and the discussion of this section is summarized in the following analogue of Theorem 4 for continuous stable population theory.

THEOREM 7. Let b(t) satisfy the renewal equation

$$b(t) = g(t) + \int_o^t \varphi(s)b(t-s)ds \quad ,$$

and let a solution of the form

$$b(t) = \sum_{i=o}^{\infty} a_i e^{r_i t}$$

exist. Then

(i) if $r < 0$,

(a) $\lim_{t\to\infty} b(t) = 0$

(b) $\int_o^\infty b(t)dt = \frac{a}{1 - N}$, where $N = \int_o^\infty \varphi(s)ds$

(c) $\lim_{t\to\infty} e^{-rt}b(t) = a$

(ii) if $r = 0$,

$$\lim_{t\to\infty} b(t) = a$$

(iii) if $r > 0$,

$$\lim_{t\to\infty} e^{-rt}b(t) = a$$

and, in addition, if for each i either $a_i = 0$ or $Re(r_i) < 0$, then $f(t) = b(t) - ae^{rt}$ exists on $(0,\infty)$, and

$$\lim_{t\to\infty} [b(t) - ae^{rt}] = 0.$$

Note that if (v,x(o)) exists, then a = (v,x(o)); a general expression for the coefficients $a_o, a_1, \ldots$ was given by Feller [1941]. All conclusions of Theorem 7 except the last follow from renewal theory; only the proof of the last depends on the existence of the Lotka solution, as will appear from the proof given here of Theorem 6.

<u>Case i: $r < 0$</u>. A bounded solution of the renewal equation (11) exists for $t \geq 0$. Conclusions (a) and (b) are always true when $r < 0$, while

conclusion (c) requires, in addition to the existence of a negative real solution to equation (12), that g satisfy $\int_0^\infty e^{-rt}g(t)dt < \infty$. Thus all results hold in demography, without recourse to the existence of the Lotka solution. For proofs and further discussion of the case $\int_0^\infty \varphi(t)dt < 1$, see Feller ([1971], pp. 374-7).

<u>Case ii: $r = 0$</u>. A solution of equation (11) exists for $t \geq 0$, and all the results of renewal theory apply. Among the weakest of these is (ii) of Theorem 7, the renewal density theorem (Smith [1958]). Thus Theorems 5 and 6 hold when $r = 0$, without the requirement of the existence of the Lotka solution; they are of course equivalent in this case. For proofs and stronger results, see Smith [1958] and Feller ([1971], Chapter XI).

<u>Case iii: $r > 0$</u>. A bounded solution of equation (11) exists for any finite interval $[0,T]$. To extend study of this solution to the non-negative real numbers, it is usual (Feller [1971], p. 377) to consider instead $b^*(t) = e^{-rt}b(t)$, which reduces this case to the preceding one; $b^*(t)$ is bounded for $t \geq 0$, and $\lim_{t\to\infty} e^{-rt}b(t) = a$. This change of emphasis is not entirely satisfactory, and stronger results are given in Theorem 6.

To prove Theorem 6, it is convenient to introduce

$$f(t) = b(t) - ae^{rt} \tag{14}$$

and the partial sums

$$f_n(t) = \sum_{i=1}^{n} a_i e^{r_i t} \quad . \tag{15}$$

By Theorem 5, $\lim_{t\to\infty} e^{-rt}f(t) = 0$, but $f(t)$ may itself behave in any of the ways illustrated in Figures 1 to 3; $f(t)$ is "stable" in the sense used in mechanics if $\lim_{t\to\infty} f(t) = 0$. When the Lotka solution (12) to equation (11) is valid,

$$f(t) = \sum_{i=1}^{\infty} a_i e^{r_i t}$$

formally, and so $\lim f_n(t)$ exists whenever $f(t)$ does. The behavior of $\{f_n(t)\}$ and of $f(t)$ are described by the following

LEMMA. Let $b(t) = \sum_{i=0}^{\infty} a_i e^{r_i t}$ be a solution of the renewal equation (11) on any finite interval, and define $f(t)$ and $f_n(t)$ by equations (14) and (15). Then if for each i either $a_i = 0$ or $Re(r_i) \leq 0$, $f(t)$ is bounded for $t \geq 0$ and is the uniform limit of the sequence of partial sums $\{f_n(t)\}$.

<u>Proof</u>: Since the Lotka solution exists, $\sum_{i=0}^{\infty} a_i$ is absolutely convergent (Feller [1941]), and so

$$\begin{aligned} |f(t)| &= \left|\sum_{i=1}^{\infty} a_i e^{r_i t}\right| \\ &\leq \sum_{i=1}^{\infty} |a_i| e^{Re(r_i)t} \\ &< \sum_{i=0}^{\infty} |a_i| \end{aligned}$$

implies that $f(t)$ is bounded for $t \geq 0$ under the conditions of the Lemma; the last inequality follows from $a_o \neq 0$. To establish uniform convergence of $\{f_n(t)\}$ to $f(t)$,

$$\begin{aligned} |f(t) - f_n(t)| &= \left|\sum_{i=n+1}^{\infty} a_i e^{r_i t}\right| \\ &\leq \sum_{i=n+1}^{\infty} |a_i| \quad , \end{aligned}$$

where again $a_i = 0$ if $Re(r_i) > 0$, and so

$$\lim_{n \to \infty} f_n(t) = f(t)$$

uniformly in t follows from absolute convergence of $\{f_n(o)\}$ to $f(o)$. Under the conditions of the Lemma, $f(t)$ either remains bounded or converges; the latter possibility is described by Theorem 6.

<u>Proof of Theorem 6</u>: Note the stronger restriction on $Re(r_i)$ in the Theorem than in the Lemma. Because of the uniform convergence of f(t), limiting operations may be interchanged in

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \lim_{n\to\infty} f_n(t)$$

to give

$$\lim_{t\to\infty} f(t) = \lim_{n\to\infty} \lim_{t\to\infty} f_n(t)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \lim_{t\to\infty} a_i e^{r_i t}$$

$$= 0 \quad ,$$

where the last equality follows, when $a_i \neq 0$, from

$$\lim_{t\to\infty} |e^{r_i t}| = \lim_{t\to\infty} e^{Re(r_i)t} = 0$$

whenever $Re(r_i) < 0$. Necessary conditions for $\lim_{t\to\infty} f(t) = 0$ have not been investigated.

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Appendix

Because no projection matrix with $\max_{\lambda_j \neq r} |\lambda_j| \geq 1$ could be found for human populations, an indirect procedure was used in the preparation of Figures 1 to 3. The three net maternity functions used each have the same underlying fertility pattern by age, in the sense that $\{x_t/r^t\}$ (and hence $\{e_t/r^t\}$) converges identically in each case. This behavior, which is not illustrated, is similar to that shown in Figure 1, but with faster convergence. Calculations were based on the following argument (for details, see Sykes [1973]), using n = 10 five-year age classes.

From equation (7), with $u_1 = 1$,

$$e_{t1} = x_{t1} - r^t(v, x_o) \quad , \tag{A1}$$

and from equation (2)

$$x_{t1} = \sum_{i=1}^{10} b_i x_{t-1,i}$$

$$= \sum_{i=1}^{10} \varphi_i x_{t-1,i}/p_i$$

where $\varphi_i = p_i b_i$. Noting that $x_{t-1,i} = p_i x_{t-i,1}$ for $t \geq 10$,

$$x_{t1} = \sum_{i=1}^{10} \varphi_i x_{t-i,1} \tag{A2}$$

is valid at least for $t \geq 10$, and can be made generally so by an appropriate choice of initial conditions. Introducing $\eta_t = r^{-t} x_{t1}$ and $\beta_i = r^{-i}\varphi_i$, equation (A2) becomes

$$\eta_t = \sum_{i=1}^{10} \beta_i \eta_{t-i} \tag{A3}$$

so that

$$e_{t1} = r^t\left[\eta_t - (v, x_o)\right] \quad . \tag{A4}$$

Finally, if the initial age distribution is $x_o' = k\ (1,0,\ldots,0)$, then $(v, x_o) = k/\mu$ is invariant for all matrices with the same value of $\beta = (\beta_1, \ldots \beta_{10})$.

Here, the value of β selected was that of Japan, 1964 (Keyfitz and Flieger [1971], pp. 396-397), and x_o was taken as $x_o' = 5.56140\ (1,0,\ldots,0)$ to make $(v,x_o) = 1$. Three values of r were chosen to give the desired results, and equation (A4) was used in the calculations; the values of the net maternity functions implied by these choices are shown in the table below. Note that the eigenvalues of the matrix with net maternity function φ given by $\varphi_i = r^i \beta_i$ are just r times those of a matrix with net maternity function β, and so the various maximal eigenvalues other than r are the product of the appropriate r and 0.89222, the value of the maximal eigenvalue other than 1 for a matrix with the β of Japan, 1964. Note also that any value of $p = (p_1,\ldots,p_n)$ could be selected to form a population projection matrix corresponding to a given value of φ; given r and β, the convergence of $\{e_t\}$ is independent of mortality.

The sequence $\{e_{t1}\}$ was calculated and plotted for each Figure using a Hewlett-Packard model HP-9821 calculator with model 9862 plotter.

Table

Values of the Net Maternity Function φ Used in Calculations

for Figures 1-3. ($\varphi_i = p_i b_i = r^i \beta_i$ in each column)

i	β_i $r=1$	Figure 1 ($r_1=1.08$)	Figure 2 ($r_2=1.120797$)	Figure 3 ($r_3=1.14$)
1	0.00000	0.00000	0.00000	0.00000
2	0.00000	0.00000	0.00000	0.00000
3	.00406	.00511	.00572	.00602
4	.12776	.17382	.20161	.21578
5	.36532	.53677	.64611	.70339
6	.34493	.54736	.68374	.75711
7	.12618	.21625	.28034	.31574
8	.02725	.05044	.06786	.07773
9	.00426	.00852	.01189	.01385
10	.00024	.00052	.00075	.00089
N	1.00000	1.53879	1.89802	2.09051

Figures 1-3. Behavior of $e_{t1} = x_{t1} - r^t(v,x_o)$ for $t = 1,2,\ldots,100$, with $(v,x_o) = 1$ and values of r chosen so that $\max\limits_{\lambda_j \neq r} |\lambda_j| \lesseqgtr 1$.

Figure 1. $\max |\lambda_j| < 1$; e_{t1} converges to zero
($r = 1.08$, $\max |\lambda_j| = .9636$)

Figure 2. $\max |\lambda_j| = 1$; e_{t1} bounded, but oscillates
($r = 1.120797$, $\max |\lambda_j| = 1$)

Figure 3. $\max |\lambda_j| > 1$; e_{t1} diverges and is unbounded
($r = 1.14$, $\max |\lambda_j| = 1.0171$)

FIGURE 1

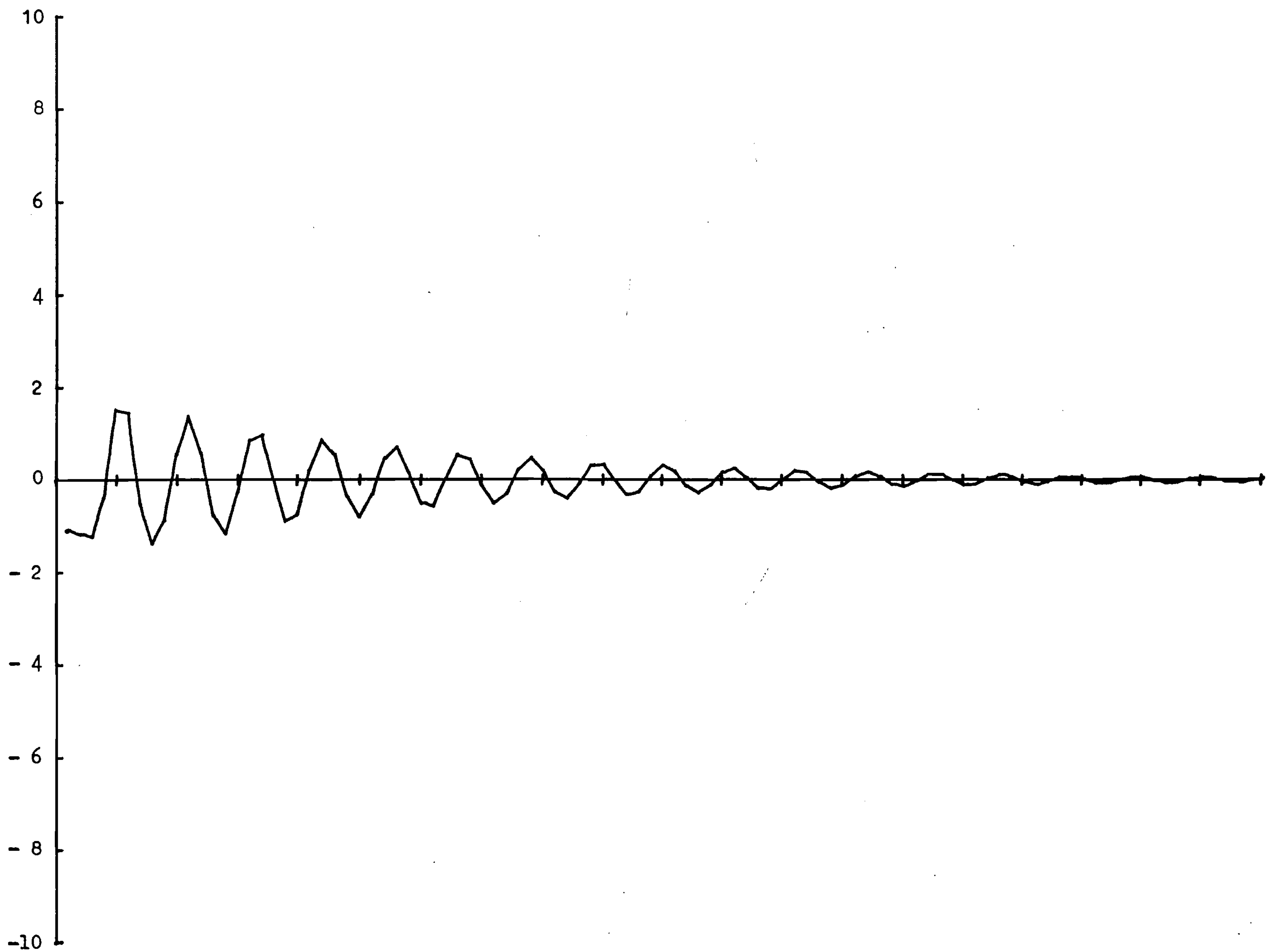

FIGURE 2

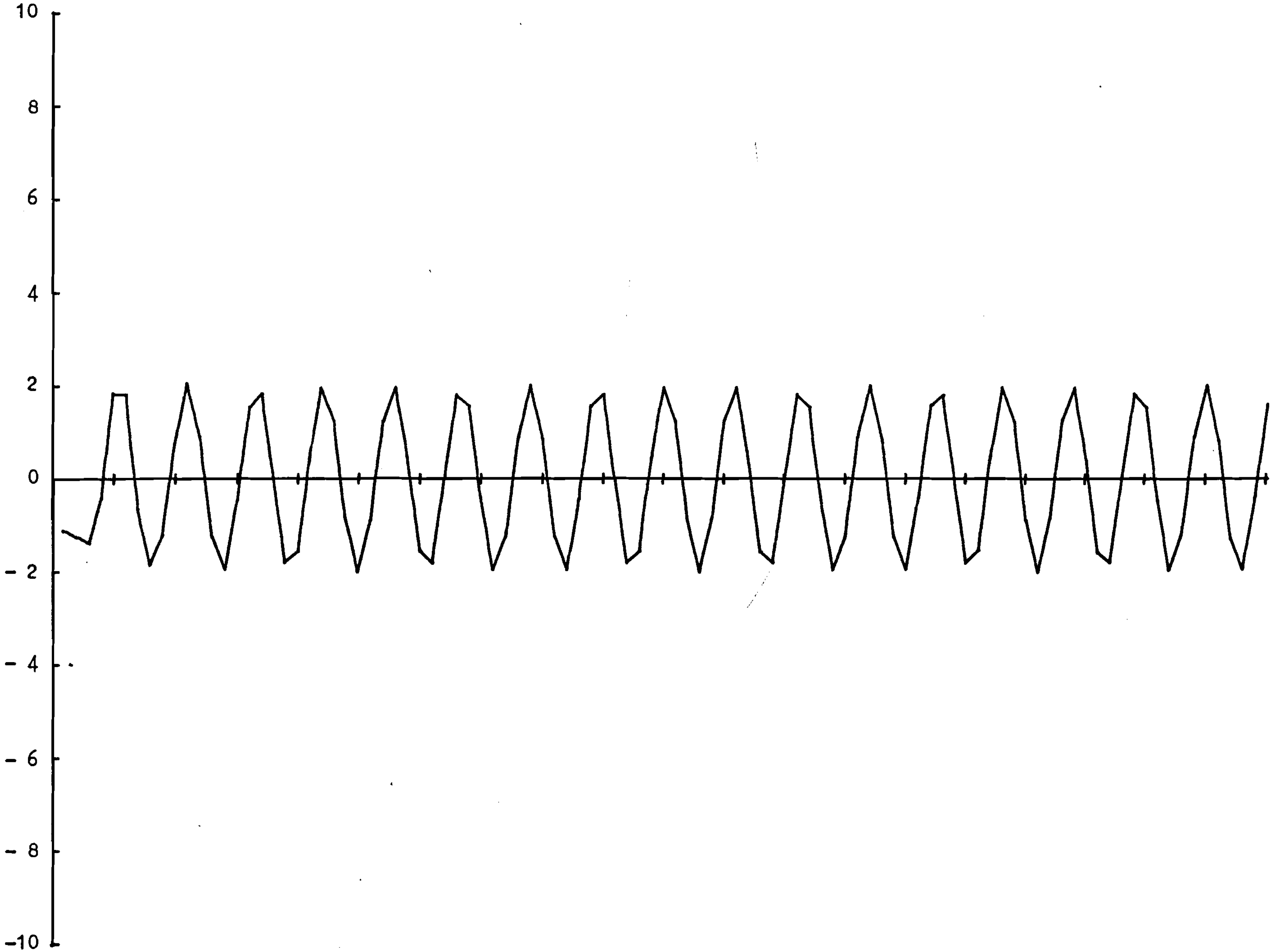

FIGURE 3