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ON UNIQUENESS QUESTIONS FOR HYPERBOLIC DIFFERENTIAL EQUATIONS

by

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ABSTRACT

This note is concerned with questions of uniqueness, existence and convergence of successive approximations for a solution of an initial value problem, where $z_{xy} = f(x,y,z,z_x,z_y)$ and $z(x,0)$, $z(0,y)$ are assigned. There are obtained analogues of the Nagumo and Kamke criteria in the theory of ordinary differential equations. The method employed is related to the arguments used by Viswanatham to prove the convergence of successive approximations for ordinary differential equations under conditions similar to those in Kamke's general uniqueness theorem.

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1. Statement of results. This note is concerned with the existence and uniqueness of solutions of the initial value problem

$$z_{xy} = f(x,y,z,p,q), z(x,0) = \sigma(x), z(0,y) = \tau(y),$$
$$\text{where } \sigma(0) = \tau(0) = z_0$$

on a rectangle $R : 0 \leqq x \leqq a,\ 0 \leqq y \leqq b$. By a solution is meant a continuous function having partial derivatives almost everywhere and satisfying the integral equation

$$z(x,y) = \sigma(x) + \tau(y) - z_0 + \int_0^x \int_0^y f(s,t,z(s,t),z_x(s,t), z_y(s,t))dsdt. \tag{1}$$

Actually it will be clear from the conditions imposed on σ, τ and f that any solution of (1) is uniformly Lipschitz continuous. Let D be the five-dimensional set $D = \{(x,y,z,p,q) : (x,y) \in R \text{ and } z,p,q \text{ arbitrary}\}$. Let $f(x,y,z,p,q)$ be defined and continuous on D, such that $|f(x,y,z,p,q)| < N = \text{const.}$ for $(x,y,z,p,q) \in D$. Let $\sigma(x)$, $\tau(y)$ be defined and uniformly Lipschitz continuous on $0 \leqq x \leqq a$, $0 \leqq y \leqq b$, respectively (so that $|\sigma(x) - \sigma(\bar{x})| \leqq K|x - \bar{x}|$, $|\tau(y) - \tau(\bar{y})| \leqq K|y - \bar{y}|$ for some

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constant K) and let $\sigma(0) = \tau(0) = z_0$. In addition, for $(x,y) \in R$ and arbitrary $z,p,q,\bar{z},\bar{p},\bar{q}$ assume that

(2) $$|f(x,y,z,p,q) - f(x,y,\bar{z},\bar{p},\bar{q})| \leqq \varphi(x,y,|z - \bar{z}|,|p - \bar{p}|,|q - \bar{q}|),$$

where $\varphi(x,y,z,p,q)$ is a continuous, non-negative function defined for $(x,y) \in R$ and non-negative z,p,q, non-decreasing in each of the variables z,p,q with the property that for every (α,β), where $0 < \alpha \leqq a$, $0 < \beta \leqq b$, the only solution of

(3) $$z(x,y) = \int_0^x \int_0^y \varphi(s,t,z(s,t),z_x(s,t),z_y(s,t))dsdt$$

in the rectangle $R_{\alpha\beta}$: $0 \leqq x \leqq \alpha$, $0 \leqq y \leqq \beta$ is $z \equiv 0$.

Theorem (*). <u>Under the above assumptions on</u> σ, τ, f <u>and</u> φ, (1) <u>possesses one and only one solution on</u> R. <u>This solution is the uniform limit of the successive approximations defined by</u>

(4_0) $$z_0(x,y) = \sigma(x) + \tau(y) - z_0$$

<u>and, for</u> n = 1,2,3,..., <u>by</u>

(4_n) $$z_n(x,y) = z_0(x,y) + \int_0^x \int_0^y f(x,y,z_{n-1}(s,t),z_{n-1\,x}(s,t), z_{n-1\,y}(s,t))dsdt.$$

The existence assertion of (*) neither implies nor is implied by that in Hartman-Wintner [3] and its generalizations due to Conti, Szmydt, Ciliberto, Kisynski (for references, see [6] and [2]). The uniqueness assertion of (*) can be considered as a crude analogue of Kamke's uniqueness theorem (cf. [5], p. 139) in the theory of ordinary differential equations. Finally, the assertion concerning the convergence of successive approximations is an analogue of a result on ordinary differential equations (cf. Viswanatham [8] and references there to van Kampen, to Wintner and to Dieudonné, and Coddington and Levinson [1]).

A theorem similar to (*), in which f and φ do not depend on p,q is proved by Guglielmino [2]. The proof of (*) below will be a generalization of that of [2]. A uniqueness theorem for (1) involving a majorant function of the form $\varphi(z,p,q) = \varphi(|z| + |p| + |q|)$ is given in [6].

Remark. It will be clear from the proofs that (*) *remains valid if* f,z,p,q,σ,τ *are n-vectors* (say, with the norm $|z| = \sum_{k=1}^{n} |z^k|$ or $|z| = \max(|z^1|,\ldots,|z^n|)$ if $z = (z^1,\ldots,z^n)$).

A theorem suggested by Nagumo's uniqueness theorem (cf. [4], p. 97) for ordinary differential equations is the following:

Theorem (**). *Let* $f(x,y,z,p,q)$ *be defined, continuous and bounded on* D, *and satisfy, for* $xy > 0$ *and arbitrary* $z,p,q,\bar{z},\bar{p},\bar{q}$,

$$(5)\qquad |f(x,y,z,p,q) - f(x,y,\bar{z},\bar{p},\bar{q})| \leqq c_1(x,y)|z - \bar{z}|/xy + c_2(x,y)|p - \bar{p}| /y + c_3(x,y)|q - \bar{q}|/x,$$

where $c_i(x,y)$, $i = 1,2,3$, *are non-negative, continuous functions such that*

$$c_1 + c_2 + c_3 \equiv 1.$$

Let $\sigma(x)$, $\tau(y)$ *be as in* (*), *and, in addition, let*

$$(6)\qquad \sigma_x(+0) = \lim_{x \to +0} \sigma_x(x), \quad \tau_y(+0) = \lim_{y \to +0} \tau_y(y)$$

exist. Then (1) *has at most one solution* $z = z(x,y)$. *Furthermore, if*

$$(6^*)\qquad c_1(0,0) > 0,$$

then the solution is the uniform limit of the successive approximations (4).

In (6), x [or y] tends to +0 through the set of values on which σ_x [or τ_y] exists.

Remark 1. (**) is valid if f,z,p,q,σ,τ are n-vectors (say $z = (z^1,\ldots,z^n)$ and either $|z| = \sum_{k=1}^{n} |z^k|$ or $|z| = \max(|z^1|,\ldots,|z^n|)$).

Remark 2. A modification of an example of Perron [7] in the theory of ordinary differential equations will show that (**) is false if

c_1 = const. > 1, $c_2 \equiv c_3 \equiv 0$ (so that f does not depend on p,q). Also, a modification of an example of Haviland [4] shows that successive approximations need not converge if c_1 = const. > 1, $c_2 = c_3 \equiv 0$.

The proof of (*) will be given in Sections 2-4 below; that of (**) in Sections 5-6; finally, the proof of the last remark will be indicated in Section 7.

The results above answer some questions suggested by Professor P. Hartman. I also wish to acknowledge helpful discussions with him.

2. Proof of (*). Preliminaries. In the proof of (*) below, there is no loss of generality in supposing that φ is bounded, say $0 \leqq \varphi(x,y,z,p,q) \leqq 2N$ on D. For otherwise φ can be replaced by $\bar{\varphi}$, where $\bar{\varphi}(x,y,z,p,q)$ equals $\varphi(x,y,z,p,q)$ or 2N according as $\varphi(x,y,z,p,q)$ does or does not exceed 2N. It is clear that $\bar{\varphi}$ is continuous and non-decreasing in each of the variables z,p,q. Furthermore, the only solution z(x,y) of

(3') $$z(x,y) = \int_0^x \int_0^y \bar{\varphi}(s,t,x(s,t),z_x(s,t),z_y(s,t))dsdt$$

on any rectangle $R_{\alpha\beta}$: $0 \leqq x \leqq \alpha$ ($\leqq a$), $0 \leqq y \leqq \beta$ ($\leqq b$) is $z \equiv 0$.

In order to see this, note that $\varphi(x,y,0,0,0,) \equiv 0$, so that there exists an $\epsilon > 0$ such that $0 \leqq \varphi(x,y,z,p,q) \leqq 2N$ if $|z|$, $|p|$, $|q| < \epsilon$. Suppose that $z(x,y) \not\equiv 0$ is a solution of (3') on $R_{\alpha\beta}$. Let d, $0 \leqq d \leqq (\alpha^2 + \beta^2)^{\frac{1}{2}}$, be the largest value of r for which $z(x,y) \equiv 0$ in the intersection S_r of $x^2 + y^2 \leqq r^2$ and $R_{\alpha\beta}$. If U is any neighborhood of S_d (relative to $R_{\alpha\beta}$), there exists a rectangle $R_{\gamma\delta}$ in U on which $z \not\equiv 0$. Since $z \equiv 0$ on S_d, it is clear that if U is "sufficiently small", then, on U (hence on $R_{\gamma\delta}$), $|z| < \epsilon$ and, almost everywhere, $|z_x| + |z_y| < \epsilon$. But then $z \not\equiv 0$ is a solution of (3) on $R_{\gamma\delta}$. Since this is impossible, the only solution of (3') on $R_{\alpha\beta}$ is $z \equiv 0$.

It will be convenient to have the following notation. R_1 denotes a subset (not always the same) of R of the form E x [0,b], where E is a (Lebesgue) measurable subset of [0,a] with meas E = a. Similarly, R_2 is a subset (not always the same) of the form [0,a] x E, where E is a measurable subset of [0,b] and meas E = b. Partial derivatives z_x, z_y of a function z will be denoted by p,q.

3. Lemma for (*). The proof of (*) will depend on the following lemma.

Lemma 1. *Let $\alpha(x,y)$, $\beta(x,y)$ $\gamma(x,y)$ be non-negative, measurable functions defined on R, R_1, R_2, respectively, such that α is continuous, β is uniformly Lipschitz continuous with respect to y and γ is uniformly Lipschitz continuous with respect to x. In addition, let*

$$(7)\qquad \alpha(x,y) \leqq \int_0^x\int_0^y \varphi(s,t,\alpha(s,t),\beta(s,t),\gamma(s,t))dsdt,$$

$$(8)\qquad \beta(x,y) \leqq \int_0^y \varphi(s,t,\alpha(x,t),\beta(x,t),\gamma(x,t))dt,$$

$$(9)\qquad \gamma(x,y) \leqq \int_0^x \varphi(s,y,\alpha(s,y),\beta(s,y),\gamma(s,y))ds,$$

where φ satisfies the conditions of (*) and is bounded. Then $\alpha \equiv \beta \equiv \gamma \equiv 0$.

Note that the Lipschitz continuity of β [or α] with respect to y [or x] is assumed to be uniform with respect to x and y.

The proof of the lemma below follows a suggestion made by R. Sacksteder. My original proof, which will be omitted, depended on two results. The first result is an existence theorem for

$$(10)\qquad z(x,y) = \Psi(x,y) + \int_0^x\int_0^y \varphi(s,t,z(s,t),p(s,t),q(s,t))dsdt,$$

where Ψ is a non-negative, uniformly Lipschitz continuous function which is non-decreasing in x and in y. This existence theorem is proved by using the successive approximations $z_0 = \Psi(x,y)$ and

$$(11)\qquad z_n(x,y) = z_0(x,y) + \int_0^x\int_0^y \varphi(s,t,z_{n-1},p_{n-1},q_{n-1})dsdt$$

which satisfy

(12) $\quad z_n \leqq z_{n+1},\ p_n \leqq p_{n+1},\ q_n \leqq q_{n+1}.$

The second result is the fact that if ψ is replaced by another function $\bar{\psi}$ with similar properties and

(13) $\quad \psi \leqq \bar{\psi},\ \psi_x \leqq \bar{\psi}_x,\ \psi_y \leqq \bar{\psi}_y,$

then the corresponding solution $\bar{z}$ satisfies

(14) $\quad z \leqq \bar{z},\ p \leqq \bar{p},\ q \leqq \bar{q}.$

Proof. Define sequences of successive approximations as follows: Let

(15) $\quad z_0(x,y) = \alpha(x,y),\ u_0(x,y) = \beta(x,y),\ v_0(x,y) = \gamma(x,y)$

and, for $n \geqq 1$,

(16) $\quad z_n(x,y) = \int_0^x \int_0^y \varphi(s,t,z_{n-1}(s,t),u_{n-1}(s,t),v_{n-1}(s,t))dsdt,$

(17) $\quad u_n(x,y) = \int_0^y \varphi(x,t,z_{n-1}(x,t),u_{n-1}(x,t),v_{n-1}(x,t))dt,$

(18) $\quad v_n(x,y) = \int_0^x \varphi(s,y,z_{n-1}(s,y),u_{n-1}(s,y),v_{n-1}(s,y))ds.$

The functions z_n, u_n, v_n are defined on sets R, R_1, R_2, respectively, which can be taken independent of n. The inequalities (7), (8), (9) give the case $n = 0$ of

(19) $\quad z_n \leqq z_{n+1},\ u_n \leqq u_{n+1},\ v_n \leqq v_{n+1}.$

The cases $n > 0$ of these inequalities follow by induction by virtue of the monotony of φ.

The boundedness of φ implies the uniform boundedness of the functions z_n, u_n, v_n. Hence, as $n \to \infty$

(20) $\quad z = \lim z_n,\ u = \lim u_n,\ v = \lim v_n,$

exist on R, R_1, R_2, respectively. It is clear from (15) and (19), (20) that

(21) $\quad 0 \leqq \alpha \leqq z,\ 0 \leqq \beta \leqq u,\ 0 \leqq \gamma \leqq v.$

Lebesgue's theorem on term-by-term integration under bounded convergence implies

$$(22) \qquad z(x,y) = \int_0^x \int_0^y \varphi(s,t,z(s,t),u(s,t),v(s,t))dsdt,$$

$$(23) \qquad u(x,y) = \int_0^y \varphi(x,t,z(x,t),u(x,t),v(x,t))dt,$$

$$(24) \qquad v(x,y) = \int_0^x \varphi(s,y,z(s,y),u(s,y),v(s,y))ds.$$

It is clear that $z_x = u$, $z_y = v$ almost everywhere. Thus the assumption on φ concerning (6) shows that $z \equiv u \equiv v \equiv 0$. Lemma 1 follows from (21).

4. Proof of (*). (i). Let $z(x,y)$ be a solution of (1). There exist functions $u(x,y)$, $v(x,y)$ defined on sets R_1, R_2, respectively, such that

$$(25) \qquad z(x,y) = \sigma(x) + \tau(y) - z_0 + \int_0^x \int_0^y f(s,t,z(s,t),u(s,t), v(s,t)dsdt,$$

$$(26) \qquad u(x,y) = \sigma_x(x) + \int_0^y f(x,t,z(x,t),u(x,t),v(x,t))dt,$$

$$(27) \qquad v(x,y) = \tau_y(y) + \int_0^x f(s,y,z(s,y),u(s,y),v(s,y))ds,$$

and the relations $u = z_n$ and $v = z_y$ hold almost everywhere. In order to see this, note that almost everywhere on R,

$$z_x(x,y) = \sigma_x(x) + \int_0^y f(x,t,z(x,t),u(x,t),v(x,t))dt,$$

$$z_y(x,y) = \sigma_y(y) + \int_0^x f(s,y,z(s,y),u(s,y),v(s,y))ds.$$

The expressions on the right side of these equations are defined for (x,y) on sets R_1, R_2, respectively. Define $u(x,y)$, $v(x,y)$ to be these expressions on R_1, R_2. In particular $z_x = u$ and $z_y = v$ almost everywhere. Hence (26), (27) hold on (possibly different) sets R_1, R_2. Clearly (25) is valid for all (x,y) on R.

(ii). Uniqueness in (*). Suppose that (1) possesses two solutions $z = z_1(x,y)$, $z_2(x,y)$ on R. Let $u_1(x,y)$, $v_1(x,y)$ and $u_2(x,y)$, $v_2(x,y)$ be the functions associated with z_1, z_2 by (i). Let $\alpha = |z_1 - z_2|$, $\beta = |u_1 - u_2|$, $\gamma = |v_1 - v_2|$. If the relations (25) for $z = z_1$, z_2 are substracted, it is seen that the inequality (2) for f implies (7). Similarly

(26), (27) imply (8), (9) respectively.

The functions α, β, γ satisfy the assumptions of Lemma 1. Hence the uniqueness assertion in (*) follows from Lemma 1.

(iii). Existence and successive approximations. Let $z_0(x,y)$, $z_1(x,y),\ldots$ be the successive approximations defined by (4). Corresponding to each $z_n(x,y)$, it is possible to introduce functions $u_n(x,y)$, $v_n(x,y)$ defined on sets R_1, R_2, respectively, and satisfying $u_0 = \sigma_x(x)$, $v_0 = \tau_y(y)$,

$$(28_n) \qquad z_n(x,y) = \sigma(x) + \tau(y) - z_0 + \int_0^x \int_0^y f(s,t,z_{n-1}(s,t), u_{n-1}(s,t), v_{n-1}(s,t))dsdt,$$

$$(29_n) \qquad u_n(x,y) = \sigma_x(x) + \int_0^y f(x,t,z_{n-1}(x,t),u_{n-1}(x,t),v_{n-1}(x,t))dt,$$

$$(30_n) \qquad v_n(x,y) = \tau_y(y) + \int_0^x f(s,y,z_{n-1}(x,t),u_{n-1}(s,y),v_{n-1}(x,t))ds.$$

The sets R_1, R_2 can be assumed to be independent of n.

Let $Z_{mn} = |z_m - z_n|$, $U_{mn} = |u_m - u_n|$, $V_{mn} = |v_m - v_n|$ and

$$(31) \qquad \alpha_k(x,y) = \operatorname*{l.u.b}_{m,n \geqq k} Z_{mn}, \ \beta_k(x,y) = \operatorname*{l.u.b}_{m,n \geqq k} U_{mn}, \ \gamma_k(x,y) = \operatorname*{l.u.b}_{m,n \geqq k} V_{mn}.$$

It is clear that Z_{mn}, U_{mn}, V_{mn} are uniformly Lipschitz continuous with respect to (x,y), x, y, respectively, and that a corresponding statement holds for α_k, β_k, γ_k.

By subtracting the relation (28_n) from (28_{n-1}) and using the inequality (2) for f, it is seen that

$$Z_{mn}(x,y) \leqq \int_0^x \int_0^y \varphi(s,t,Z_{m-1\,n-1}(s,t),U_{m-1\,n-1}(s,t), V_{m-1\,n-1}(s,t))dsdt.$$

Thus, if $m,n \geqq k$, the monotony of φ shows that

$$Z_{mn}(x,y) \leqq \int_0^x \int_0^y \varphi(s,t,\alpha_{k-1}(s,t),\beta_{k-1}(s,t),\gamma_{k-1}(s,t))dsdt.$$

Hence

$$\alpha_k(x,y) \leqq \int_0^x \int_0^y \varphi(s,t,\alpha_{k-1}(s,t),\beta_{k-1}(s,t),\gamma_{k-1}(s,t))dsdt.$$

Similarly

$$\beta_k(x,y) \leqq \int_0^y \varphi(x,t,\alpha_{k-1}(x,t),\beta_{k-1}(x,t),\gamma_{k-1}(x,t))dt,$$

$$\gamma_k(x,y) \leqq \int_0^x \varphi(s,y,\alpha_{k-1}(s,y),\beta_{k-1}(s,y),\gamma_{k-1}(s,y))ds.$$

By (31), the sequences $\{\alpha_k(x,y)\}$, $\{\beta_k(x,y)\}$, $\{\gamma_k(x,y)\}$ are non-increasing (and non-negative). Let $\alpha(x,y)$, $\beta(x,y)$, $\gamma(x,y)$ denote the respective limits of these sequences. The Lipschitz continuity of α_k, β_k, γ_k is preserved under the limiting process. Lebesgue's theorem on term-by-term integration under bounded convergence gives the inequalities (7), (8), (9). Hence Lemma 1 shows that $\alpha \equiv 0$, $\beta \equiv 0$, $\gamma \equiv 0$ on R, R_1, R_2, respectively. This implies the existence of the functions $z = \lim z_n$, $u = \lim u_n$, $v = \lim v_n$ on R, R_1, R_2, as $n \to \infty$, satisfying (25), (26), (27). It is clear that the limit function $z(x,y)$ is a solution of (1).

Finally, the equicontinuity of the functions $z_n(x,y)$ (implied by their uniform Lipschitz continuity) shows that $z(x,y)$ is the *uniform* limit of the $z_n(x,y)$. This proves (*).

5. *Lemma for* (**). The proof of (**) will depend on the following lemma:

Lemma 2. *Let* $\alpha(x,y)$, $\beta(x,y)$, $\gamma(x,y)$ *be non-negative, measurable functions defined on* R, R_1, R_2, *respectively, so that* α *is continuous,* β *is uniformly Lipschitz continuous with respect to* y *and* γ *is uniformly Lipschitz continuous with respect to* x. *Furthermore, assume that*

(32) $\qquad \alpha(x,y)/xy \to 0$ *as* $0 < xy \to 0$

and that, uniformly with respect to x *and* y, *respectively,*

(33) $\qquad \beta(x,y)/y \to 0$ *as* $y \to 0$ *and* $\gamma(x,y)/x \to 0$ *as* $x \to 0$.

Finally, suppose that

$$(34) \qquad \alpha(x,y) \leqq \int_0^x \int_0^y \{c_1(s,t)\alpha(s,t)/st + c_2(s,t)\beta(s,t)/t + c_3(s,t)\gamma(s,t)/s\} dsdt,$$

$$(35) \qquad \beta(x,y) \leqq \int_0^y \{c_1(x,t)\alpha(x,t)/xt + c_2(x,t)\beta(x,t)/t + c_3(x,t)\gamma(x,t)/x\} dt,$$

$$(36) \qquad \gamma(x,y) \leqq \int_0^x \{c_1(s,y)\alpha(s,y)/sy + c_2(s,y)\beta(s,y)/y + c_3(s,y)\gamma(s,y)/s\} ds,$$

where c_1, c_2, c_3 *are as in the first part of* (**). *Then* $\alpha \equiv \beta \equiv \gamma \equiv 0$.

Proof. By (32), if $\alpha(x,y)/xy$ is defined as 0 when $xy = 0$, it becomes a continuous function on R. Hence, it assumes its maximum M_1 at some point $(x^1,y^1) \in R$. Let $M_2 = \text{l.u.b. } \beta(x,y)/y$ and $M_3 = \text{l.u.b. } \gamma(x,y)/x$ for $(x,y) \in R$.

Note that there exist numbers M_{jk}, where $j,k = 1,2,3$, satisfying

$$(37) \qquad M_{jk} \geqq 0 \text{ and } \sum_{k=1}^{3} M_{jk} = 1 \quad \text{for } j = 1,2,3,$$

and

$$(38_j) \qquad M_j \leqq \sum_{k=1}^{3} M_{jk} M_k$$

If $M_1 \neq 0$, then $M_1 = \alpha(x^1,y^1)/x^1y^1$ holds for some point (x^1,y^1) of R with $x^1y^1 > 0$. In this case, (38_1) follows from (34) with $(x,y) = (x^1,y^1)$ if

$$(39) \qquad M_{1k} = (x^1y^1)^{-1} \int_0^{x^1} \int_0^{y^1} c_k(s,t)dsdt.$$

If $M_1 = 0$, let $M_{1k} = c_k(0,0)$.

In order to obtain (38_2), let (x_j,y_j), where $j = 1,2,\ldots$, be points of R such that $\lim (x_j,y_j) = (x^2,y^2)$ exists, $\lim \beta(x_j,y_j)/y_j = M_2$ and $\lim \beta(x_j,y) = \beta(y)$ exists uniformly for $0 \leqq y \leqq b$. Then (35) leads to (38_2) with

$$(40) \qquad M_{2k} = (y^2)^{-1} \int_0^{y^2} c_k(x^2,t)dt \text{ or } M_{2k} = c_k(x^2,0)$$

according as $y^2 > 0$ or $y^2 = 0$. A relation of the type (38_3) is obtained similarly.

Let $M_J = \max(M_1, M_2, M_3)$. Suppose, if possible, that $M_J > 0$. Assume, for the moment, that $M_J > M_j$ if $j \neq J$. Then, by (37) and (38_J), $M_{JJ} = 1$ and $M_{Jk} = 0$ for $k \neq J$. But the derivation of (38_J) can then be modified to obtain $M_J < M_J$. For example, if $J = 1$, then $c_1(s,t) \equiv 1$ and $c_2(s,t) = c_3(s,t) = 0$ in (34) when $(x,y) = (x^1, y^1)$, while $\alpha(s,t)/st$ is nearly zero for small st, so that one obtains $M_1 < M_1$. Or if $J = 2$, then $y^2 > 0$ and $c_1(x^2,t) = 1$, $c_2(x^2,t) = c_3(x^2,t) = 0$ for $0 \leqq t \leqq y^2$, while the relations

$$\beta(y) \leqq \int_0^y \beta(t)dt/t, \quad \beta(y^2)/y^2 = M_2$$

give $M_2 < M_2$ since $\beta(t)/t$ is nearly 0 for small t by the uniformity of the first limit relation in (33).

Similar arguments show that if two or three of the numbers M_1, M_2, M_3 are equal to $M_J > 0$, one is led to a contradiction. Hence $M_J = 0$. This proves the lemma.

6. Proof of (**). (i). Uniqueness in (**). Let $z = z_1(x,y)$, $z_2(x,y)$ be two solutions of (1) on R. Let $u_1(x,y)$, $v_1(x,y)$ and $u_2(x,y)$, $v_2(x,y)$ be the functions associated with them as in the proof of (*). Let $\alpha = |z_1 - z_2|$, $\beta = |u_1 - u_2|$, $\gamma = |v_1 - v_2|$. It will be verified that, as x (or y) $\rightarrow 0$, then except for sets of measure zero,

(41) $\qquad \alpha(x,y), \beta(x,y), \gamma(x,y) \rightarrow 0.$

Consider the case $x \rightarrow 0$. The assertions (41) concerning α and γ are clear. In order to verify assertion (41) for the function β, it will first be shown that if $z = z(x,y)$ is any solution of (1) (say, $z = z_1$ or $z = z_2$) and if $u(x,y)$, $v(x,y)$ are its associated functions, then

(42) $\qquad \lim u(x,y) = \rho(y)$, as $x \rightarrow 0$, exists uniformly in y.

To see this, let x_j, where $j = 1,2,3,\ldots$ be a sequence of x values such that $\lim x_j = 0$ and $\lim u(x_j,y) = \rho(y)$ exists uniformly as $j \to \infty$. Putting $x = x_j$ in (26) and letting $j \to \infty$, it is seen that

(43) $$\rho(y) = \sigma_x(+0) + \int_0^y f(0,t,\tau(t),\rho(t),\tau_y(t))dt.$$

We note that $\rho(y)$ is continuous. Furthermore, $\rho(y)$ does not depend on the sequence $x_1, x_2, \ldots$. Suppose that another sequence leads to a different limit $\bar{\rho}(y) \not\equiv \rho(y)$. By substituting $\bar{\rho}$ for ρ in (43), and subtracting, we get

(44) $$|\bar{\rho}(y) - \rho(y)| \leqq \int_0^y |f(0,t,\tau(t),\bar{\rho}(t),\tau_y(t)) - f(0,t,\tau(t),\rho(t),\tau_y(t))|dt.$$

Since $f, \rho, \bar{\rho}$ are continuous and $\rho(0) = \bar{\rho}(0) = \sigma_x(+0)$, the integrand of (44) can be made small by making y small. Hence

(45) $$|\bar{\rho}(y) - \rho(y)|/y \to 0, \text{ as } y \to 0.$$

By relation (5),

$$|\bar{\rho}(y) - \rho(y)|/y \leqq y^{-1} \int_0^y c_2(0,t)|\bar{\rho}(t) - \rho(t)|dt/t,$$

Using (45) as before, this leads to a contradiction. Hence $\bar{\rho} \equiv \rho$. Therefore every sequence, for which the limit in (42) exists, leads to the same limit. Hence (42) holds.

If $\lim u_1(x,y) = \rho_1(y)$ and $\lim u_2(x,y) = \rho_2(y)$, as $x \to 0$, we can repeat the above argument and obtain $\rho_1 \equiv \rho_2$. This completes the verification of (41).

We now verify assumptions (32) and (33) of Lemma 2. Consider, for example, the assertion

(46) $$\beta(x,y)/y \to 0 \text{ as } y \to 0.$$

By putting $u = u_1, u_2$ in (26) and subtracting we get

(47) $$\beta(x,y) \leqq \int_0^y |f(x,t,z_1(x,t),u_1(x,t),v_1(x,t)) - f(x,t,z_2(x,t),u_2(x,t),v_2(x,t))|dt.$$

Now the integrand of (47) can be made small, by making y small, and using (41). This proves (46). The other limits in (32) and (33) are verified similarly. The other assumptions of Lemma 2 are quite straightforward. Therefore $\alpha \equiv \beta \equiv \gamma \equiv 0$. This proves "uniqueness".

(ii). Existence and successive approximations in (**). Let $z_0(x,y)$, $z_1(x,y),\ldots,$ be the successive approximations defined by (4). Corresponding to $z_n(x,y)$ it is possible to introduce, as in the proof of (*), functions $u_n(x,y)$, $v_n(x,y)$ defined on sets R_1, R_2 (independent of n) and satisfying $u_0 = \sigma_x(x)$, $v_0 = \tau_y(y)$, (28_n), (29_n) and (30_n). Let Z_{mn}, U_{mn}, V_{mn} be defined as in the existence proof of (*) above. It will be verified that, given ϵ, there exists a $\delta(\epsilon)$ and an $N(\epsilon)$, such that

(48) $$Z_{mn}(x,y),\ U_{mn}(x,y),\ V_{mn}(x,y) < \epsilon$$

for $x < \delta(\epsilon)$ and for all $m,n > N(\epsilon)$. A similar statement will be seen to hold when x is replaced by y. The assertion (48) concerning Z_{mn}, and V_{mn} is clear. In order to verify (48) for the function U_{mn} it will first be shown that

(49) $$\lim u_n(x,y) = h_n(y), \text{ as } x \to 0, \text{ exists uniformly in } y \text{ and } n.$$

It is easily verified, by induction, that $h_n(y)$ exists uniformly in y for fixed n, where

(50_n) $$h_n(y) = \sigma_x(+0) + \int_0^y f(0,t,\tau(t),h_{n-1}(y),\tau_y(t))dt.$$

To see the uniformity in n, define

(51_n) $$\bar{z}_n(x,y) = z_n(x,y) - \sigma(x) - \tau(y) + z_0;\ \bar{u}_n(x,y) = u_n(x,y) - \sigma_x(x);$$

$$\bar{v}_n(x,y) = v_n(x,y) - \tau_y(y);$$

(52) $$g(x,t,z,p,q) = f(x,y,z + \sigma(x) + \tau(y) - z_0,\ p + \sigma_x(x),$$
$$q + \tau_y(y)).$$

For $\bar{u}_n$ we define $\bar{h}_n$ corresponding to h. Clearly g satisfies a condition analogous to (5), $\bar{u}_0(x,y) = \bar{h}_0(y) \equiv 0$, and

(53_n) $\quad \bar{u}_n(x,y) = \int_0^y g(x,t,\bar{z}_{n-1}(x,t),\bar{u}_{n-1}(x,t),\bar{v}_{n-1}(x,t))dt, \; n \geqq 1$

(54_n) $\quad \bar{h}_n(y) = \int_0^y g(0,t,0,\bar{h}_{n-1}(t),0)dt, \; n \geqq 1.$

To prove (49) it suffices to verify that

(55) $\quad \lim \bar{u}_n(x,y) = \bar{h}_n(y)$, as $x \to 0$, exists uniformly in y and n.

By subtracting (54_n) from (53_n), it is seen that

(56) $\quad |\bar{u}_n(x,y) - \bar{h}_n(y)| \leqq \int_0^y \{|g_1 - g_2| + |g_2 - g_3|\} dt$

where $g_1 = g(x,t,\bar{z}_{n-1}(x,t),\bar{u}_{n-1}(x,t),\bar{v}_{n-1}(x,t))$, $g_2 = g(0,t,0,\bar{u}_{n-1}(x,t),0)$ and $g_3 = g(0,t,0,\bar{h}_{n-1}(t),0)$. We note that, given $\epsilon > 0$, there exists a $\delta(\epsilon)$ such that $|g_1 - g_2| < \epsilon$ if $x < \delta$ for all y and n.

Hence, noting (5),

(57_n) $\quad |\bar{u}_n(x,y) - \bar{h}_n(y)| \leqq \int_0^y \{\epsilon + t^{-1} c_2(0,t)|\bar{u}_{n-1}(x,t) - \bar{h}_{n-1}(t)|\} dt.$

By continuity, because of (6*), $c_2(0,t) < 1$ for small $t > 0$. Hence there exists a number Θ, $0 < \Theta < 1$, such that

$$\int_0^y c_2(0,t)dt \leqq \Theta y \text{ for } 0 < y \leqq b.$$

A simple induction shows that

(58) $\quad |\bar{u}_n(x,y) - \bar{h}_n(y)| \leqq (1 - \Theta^n)\, \epsilon\, y/(1 - \Theta) \leqq b\, \epsilon/(1 - \Theta).$

This proves (55). Hence (49) is established.

Next we note that $h_n(y)$, $n = 0,1,2,\ldots$, are the successive approximations for the initial value problem

(59) $\quad dw/dt = F(t,w)$, $w(0) = \sigma_x(+0)$,

where $F(t,w) = f(0,t,\tau(t),w,\tau_y(t))$. Hence, by (5),

(60) $\quad |F(t,w) - F(t,\bar{w})| \leqq |w - \bar{w}|/t.$

Note that the existence of $\tau_y(+0)$ implies that $F(t,w) \to F(0,w) = f(0,0,\tau(0),w,\tau_y(+0))$ as $t \to +0$. The proof of the main theorem in [8] shows that these successive approximations converge uniformly, (60) being

is bounded, measurable and continuous in w (for almost all fixed t).

Nagumo's uniqueness condition (cf. [4], p. 97). Hence

(61) $\lim h_n(y) = h(y)$, exists uniformly in y as $n \to \infty$.

Now (61) and (49) together verify (48) for $U_{mn}(x,y)$. Hence (48) is established.

By an argument similar to that used in verifying (46) it is seen that, given $\epsilon > 0$, there exists $\delta(\epsilon)$ such that

$$(xy)^{-1} Z_{mn}(x,y) < \epsilon, \text{ for } xy < \delta(\epsilon), \text{ for } m,n > N(\epsilon)$$

$$x^{-1} U_{mn}(x,y) < \epsilon, \text{ for } x < \delta(\epsilon), \text{ for } m,n > N(\epsilon) \tag{62}$$

$$y^{-1} V_{mn}(x,y) < \epsilon, \text{ for } y < \delta(\epsilon), \text{ for } m,n > N(\epsilon).$$

Now defining α_k, β_k, γ_k as in (31), we note that we can substitute them for Z_{mn}, U_{mn}, V_{mn}, respectively, in (62) changing $m,n > N(\epsilon)$ to $k > N(\epsilon)$. Proceeding as in the analogous section of the proof of theorem (*), we conclude that α, β, γ satisfy (34), (35) and (36) and also (32) and (33). Therefore, by Lemma 2, the successive approximations converge uniformly to a solution of (1).

7. Counter-examples. (a). Let $a = b = 1$, $1 + \epsilon = \delta^2$, $\epsilon > 0$, $\delta > 1$. Let $f(x,y,z,p,q)$ be independent of p,q and defined by

$$f(x,y,z,p,q) = \begin{cases} 0 & \text{if } (x,y) \in R,\ z \leqq 0, \\ (1+\epsilon)\, z/xy & \text{if } (x,y) \in R,\ 0 < z < (xy)^{\delta}, \\ (1+\epsilon)\,(xy)^{\delta - 1} & \text{if } (x,y) \in R,\ (xy)^{\delta} \leqq z. \end{cases}$$

Then $f(x,y,z,p,q)$ is continuous and satisfies (5) for $c_1(x,y) = 1 + \epsilon$, (and $c_2 = c_3 \equiv 0$). Let $\sigma(x) = \tau(y) \equiv 0$. Then (1) has an infinity of solutions, namely, $z = c(xy)^{\delta}$, where $0 < c < 1$. (b). Let $a = b = 1$, $R^0 = \{(x,y) : 0 < x,y \leqq 1\}$, $1 + \epsilon = \delta^2$, $\epsilon > 0$, $\quad > 0$ and

$$f(x,y,z,p,q) = \begin{cases} 0 & \text{if } x = 0,\ y = 0, \\ (xy)^{\delta - 1} & \text{if } (x,y) \in R^0,\ z < 0, \\ (xy)^{\delta - 1} - (1+\epsilon)\, z/xy & \text{if } (x,y) \in R^0,\ 0 \leqq z \leqq (xy)^{\delta}, \\ -\,\epsilon\,(xy)^{\delta - 1} & \text{if } (x,y) \in R^0,\ (xy)^{\delta} < z. \end{cases}$$

Then $f(x,y,z,p,q)$ satisfies the same relation (5) as in example (a). However, in (4), $z_{2n} = 0$, $z_{2n+1} = (xy)^{\delta}/\delta^2$, so that the successive approximations (4) do not converge.

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