GENERATING TIGHT WAVELET FRAMES FROM SUMS OF SQUARES REPRESENTATIONS

by

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Abstract

We construct multivariate tight wavelet frames in several settings by using the theory of sums of squares representations for nonnegative trigonometric polynomials. This is done by way of two extension principles which allow us to translate the problem of constructing these frames to one of designing collections of trigonometric polynomials satisfying certain orthogonality and normalization conditions. We consider first the setting of dyadic dilation, and assume that the lowpass masks are constructed by the coset sum method, which lifts a univariate lowpass mask to a nonseparable multivariate lowpass mask, with several properties of the input being preserved. The existence of the necessary sums of squares representations is proved utilizing the special structure of these lowpass masks. We extend this first construction to the setting of prime dilation, focusing on the case of interpolatory input masks. We prove lower bounds on the vanishing moments of the highpass masks in these two constructions, and new results about the properties of the prime coset sum method.

In the first two settings, we use lowpass masks satisfying the sub-QMF condition, and apply the unitary extension principle to ensure that our filter banks result in tight wavelet frames. In the third setting, we use lowpass masks satisfying a generalization of this condition, which we dub the oblique sub-QMF condition. In fact, it turns out that for a fixed lowpass mask and vanishing moment recovery function, this condition is equivalent to the existence of highpass masks satisfying the oblique extension principle conditions. This allows us to construct multivariate tight wavelet frames for any lowpass mask satisfying the oblique sub-QMF condition, under some mild assumptions on the vanishing moment recovery function. To establish this equivalence, we first prove a new result on sums of squares representations for nonnegative multivariate trigonometric polynomials, which says that any such function may be written as a finite sum of squares of quotients of trigonometric polynomials.

We will also prove a generalization of the sum of squares result for matrices rather than functions, in which we show that a matrix with trigonometric polynomial entries which is positive semidefinite for all evaluations has a representation as a sum of squares of commuting symmetric matrices with rational trigonometric polynomial entries. We suspect that these sums of squares results for trigonometric polynomials and matrices with such entries will be of interest far beyond the wavelet construction community.

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Chapter 1

Introduction

Signals are a ubiquitous feature of modern life, in phone calls, internet, radio, medical devices, and weather stations, and coming in the forms of text, audio, images, video, and more. The problems of how to compress and transmit these signals, as well as to decompose them and extract useful information from them, are important, and solving them well can improve people's lives in countless ways. As time goes on, the kinds of data that we are able to capture are becoming much more complicated and higher dimensional, and finding good representations for such data is a challenging open problem.

One discovery that has been made several times in various fields is the existence of multiscale structure in different kinds of data: a song might be broken into a chorus and verses, each of which is further subdivided into certain phrases, which are themselves comprised of individual notes; a dissertation might be broken up into several chapters, each of which consists of several sections, and further subsections, paragraphs, sentences, words, and letters (plus some mathematical notation). The desire to separate the phenomena within a signal into different subsignals capturing the information at each scale resulted in the area of mathematics called wavelet analysis.

In addition to the analytical properties for which wavelets were originally designed, namely the efficient representation of certain kinds of signals, and the separation of these signals into their components at different scales (among others), it turns out that this multiscale idea imparts a great deal of algebraic structure into wavelet systems. This structure can be leveraged to obtain efficient coding schemes [59], and serves as a tool for constructing wavelet systems having special properties, like symmetry, regularity, and directionality.

Even after translating the problem of constructing wavelet systems from one of analysis, where we are essentially forced to work in an infinite-dimensional space of functions, to one of algebra, where we can restrict our attention to finite objects, in the form of trigonometric polynomials, it is still quite difficult to construct wavelet systems. This difficulty is exacerbated further when we impose special properties on the constructed wavelet system, or when the system is meant to analyze higher-dimensional signals.

Difficulties aside, in the univariate case, nearly every kind of wavelet system one could ask for has been constructed, or else there is some known method for finding one (see [14, 15], and references within). While it is possible to analyze multidimensional signals using the tensor product of univariate wavelet systems, this essentially means analyzing the multidimensional signal along the coordinate axes, which may do a poor job of capturing multidimensional structures in the signal, like a curved edge in an image [16]. But if we want truly multivariate wavelet systems, the existing work is fairly limited. There have been several constructions which work only in two dimensions, for example, [28, 32, 46], but those which work in higher dimensions tend to have some drawbacks. Some give conditions under which the generated wavelet system is a tight wavelet frame, without giving much insight on how to find masks satisfying those conditions, e.g., [11]. Others are focused on constructing families of multidimensional tight wavelet frames for specific theoretical goals, e.g. [27, 30]. There have been some exceptions to this trend: In [12, 31], they construct tight wavelet frames based on box splines in any dimension; and in [13], which considers the case of nonnegative lowpass filter coefficients (though all three of these papers have many contributions beyond those just mentioned). Even in these, only [31] discusses constructions with more than one vanishing moment, which is a commonly desired property for wavelet systems.

In the current work, we will construct multivariate tight wavelet frames which do not come from the tensor product, and many of which exhibit several kinds of symmetry. We have made an effort to provide general construction methods which can still be implemented by hand or with a computer. In Chapters 2 and 3, we give constructions which take in some components of a univariate wavelet system, and use them to get a nonseparable multidimensional wavelet system which shares many of the properties of the input, allowing one to work in the univariate setting where far more is known in order to get a multidimensional wavelet system which would typically be quite hard to construct. In Chapter 5, we give a very general construction for obtaining tight wavelet frames with better analytical properties, but we give several examples, and describe one way this can be implemented for box splines in any dimension. The justifications for some of the results in Chapter 5 require some technical machinery, which we develop in Chapter 4. Still, we present examples to elucidate the theory, with the hope that others clearly understand these ideas and build on them. Throughout, we seek to make the construction of multivariate tight wavelet systems having special properties more tractable and better understood, so that the full power of these tools may be leveraged to solve humanity's problems.

1.1 Overview

The following gives a brief description of each of the chapters.

In Chapter 1, we review ideas and terminology which will be needed throughout the rest of this work. This broadly falls into six categories: function spaces and wavelet systems; filters and masks in the context of wavelets; the coset sum and prime coset sum methods; extension principles; sums of squares representations; and finally, the connection between sums of squares representations and extension principles. Nearly all of this material is review from the literature, but we include the proofs of some results which are enlightening or use techniques we will employ later. We also give examples in a few places to help clarify some of these ideas, and in Section 1.7.2, we prove some new results which fit well with the material being discussed.

In Chapter 2, we consider dyadic dilation constructions based on the unitary extension principle, using lowpass masks arising from the coset sum method, which takes a univariate lowpass mask and constructs a nonseparable multivariate lowpass mask preserving several of its properties. These constructions are split into the cases of interpolatory and non-interpolatory inputs, where we use different methods for finding the sums of hermitian squares representations needed for our construction.

In Chapter 3, we generalize the previous constructions to the case of prime dilation, using the prime coset sum method. In this chapter, we focus on the case of interpolatory input masks, since finding the needed sums of squares representations in this setting uses several new conceptual ideas. We also further investigate the properties of the prime coset sum method, especially the relationships between the flatness and accuracy numbers of the input and output lowpass masks.

In Chapter 4, we prove a few new results about the existence of sums of squares representations in the context of nonnegative multivariate trigonometric polynomials, matrices with polynomial entries, and matrices with trigonometric polynomial entries. In the first case, these results are achieved using a special rational map between the complex unit circle (minus a point) and the real numbers, where the desired representations are known to hold, which we then carry back to the original domain. After generalizing some known results for matrix sums of squares representations on real space using the theory of formally real fields, we apply the previous idea to carry these results over to the trigonometric polynomial case also.

In Chapter 5, we apply the new results about sums of squares representations for multivariate trigonometric polynomials to prove an equivalence between the existence of highpass masks satisfying the oblique extension principle conditions (meaning that they generate a tight wavelet frame), and the nonnegativity of a certain trigonometric polynomial combining the lowpass mask and vanishing moment recovery function.

We will denote the rational numbers by $\mathbb{Q}$, the real numbers by $\mathbb{R}$, the complex numbers by $\mathbb{C}$, the interval $[-\pi, \pi]$ by $\mathbb{T}$, the natural numbers by $\mathbb{N} = \{0, 1, 2, \ldots\}$, and the integers by $\mathbb{Z}$. We denote the Euclidean norm on $\mathbb{C}^n$ by $\|x\|_2$ for all $x \in \mathbb{C}^n$, and we will use the notation $M_r(k)$ to denote the space of $r \times r$ matrices with entries in the ring k, for a positive integer r.

1.2 Function Spaces and Wavelet Systems

In this section, we start by defining the space $L^2(\mathbb{R}^n)$ and the Fourier transform on this space. We then discuss special systems of functions in $L^2(\mathbb{R}^n)$, before defining multiresolution analyses, and wavelet systems in this space.

1.2.1 L^p Spaces

The two spaces we will mostly be considering in this work are $L^2(\mathbb{R}^n)$ and $\ell^2(X)$, for some countable set X, but we will occasionally make use of other L^p and ℓ^p spaces, so we briefly review these spaces here.

Following [53, 54], for $1 \leq p \leq \infty$, we define $L^p(\mathbb{R}^n)$ as the vector space of all equivalence classes of Lebesgue-measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ under the equivalence relation $f \sim g$ if and only if $\|f - g\|_{L^p(\mathbb{R}^n)} = 0$, where

$$\|f\|_{L^p(\mathbb{R}^n)} = \begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x\right)^{1/p} & \text{when } p < \infty, \text{ and} \\ \operatorname{ess\,sup}_{\mathbb{R}^n} |f(x)| & \text{when } p = \infty, \end{cases}$$

where $\int$ is the Lebesgue integral. Our primary concern will be with the cases $p = 1, 2$, and ∞. These spaces are *complete* in the sense that if $(f_k)_{k=1}^{\infty} \subset L^p(\mathbb{R}^n)$ is a Cauchy sequence, then it converges to an element of $L^p(\mathbb{R}^n)$. When $1 \leq p < \infty$, $L^p(\mathbb{R}^n)$ is also *separable*, so there is a countable set of functions $\{f_k\}_{k=1}^{\infty} \subset L^p(\mathbb{R}^n)$, the finite linear combinations of which form a dense subset of $L^p(\mathbb{R}^n)$.

When $p = 2$, $L^2(\mathbb{R}^n)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\mathrm{d}x.$$

Among many other things, this means that the Cauchy-Schwartz inequality holds, so that $|\langle f, g\rangle_{L^2(\mathbb{R}^n)}| \leq \|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}$. This is a special case of Hölder's inequality, which holds for $p, q \in [1, \infty]$ related by $p^{-1} + q^{-1} = 1$ [54]: For all $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, $fg \in L^1(\mathbb{R}^n)$ and

$$\|fg\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}.$$

We will at times also consider the spaces $\ell^p(X)$, where X is some countable set. In this case, the norm is given by $\|f\|_{\ell^p(X)} = \left(\sum_{x\in X}|f(x)|^p\right)^{1/p}$ (or $\sup_{x\in X}|f(x)|$ when $p=\infty$), and when $p=2$, this is again a Hilbert space with inner product $\langle f,g\rangle_{\ell^2(X)} = \sum_{x\in X} f(x)\overline{g(x)}$.

1.2.2 The Fourier Transform on $L^2(\mathbb{R}^n)$

Let $\mathcal{S}\subset L^2(\mathbb{R}^n)$ be the subspace of infinitely differentiable functions with $\|x^\alpha D^\beta f\|_{L^\infty(\mathbb{R}^n)} < +\infty$ for all multiindices α,β (see Section 1.3.4). The Fourier transform is an operator $\mathcal{F}_0:\mathcal{S}\to\mathcal{S}$ defined by the equation

$$\mathcal{F}_0(f)(\omega) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\omega}\mathrm{d}x \text{ for all } \omega\in\mathbb{R}^n.$$

The introduction of the space $\mathcal{S}$ is useful because the evaluation maps $f(x)$ and $\mathcal{F}_0(f)(\omega)$ are well-defined in this context, but this transformation can be extended uniquely to all of $L^2(\mathbb{R}^n)$, where it is called the *Fourier transform on* $L^2(\mathbb{R}^n)$. In [53, Ch. 5, Sec. 1], this is carried out (though they scale things slightly differently there), giving a unitary transform $\mathcal{F}: L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$. In particular,

$$\|\mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} \text{ for all } f\in L^2(\mathbb{R}^n),$$

and the inverse mapping (on $\mathcal{S}$, where the evaluation maps make sense) is given by

$$f(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\mathcal{F}(f)(\omega)e^{ix\cdot\omega}\mathrm{d}\omega.$$

Typically, we will denote $\mathcal{F}(f)$ with the more compact $\hat{f}$, for $f\in L^2(\mathbb{R}^n)$, though we still use $\mathcal{F}(f)$ when convenient. We have the following result.

Result 1.1 (Properties of the Fourier Transform [52])**.** *Let $f\in\mathcal{S}$. Then the following properties hold:*

(a) $\mathcal{F}(f(\cdot+h))(\omega) = \hat{f}(\omega)e^{i\omega\cdot h}$ for any $h\in\mathbb{R}^n$.

(b) $\mathcal{F}(f(\delta\cdot))(\omega) = \delta^{-n}\hat{f}(\delta^{-1}\omega)$ for any $\delta>0$.

(c) $\mathcal{F}(f(A\cdot))(\omega) = \frac{1}{|\det(A)|}\hat{f}(A^{-T}\omega)$ *for any* $A \in M_n(\mathbb{R})$ *with* $\det(A) \neq 0$.

(d) $\mathcal{F}(D^\alpha f)(\omega) = (i\omega)^\alpha \hat{f}(\omega)$ *for any multiindex* α *(see Section 1.3.4).*

(c) is not shown in [52], but its proof is similar to the ones there.

It is easy to show that when $f, g \in L^2(\mathbb{R}^n)$, $\mathcal{F}(f * g) = (2\pi)^{n/2}\hat{f}\hat{g}$, for any $f, g \in L^2(\mathbb{R}^n)$, and this is proved for $f, g \in \mathcal{S}$ when $n = 1$ in [52, Ch. 5, Prop. 1.11] (again noting their different choice of scaling).

1.2.3 Bases and Special Systems of Functions

We follow [14, 15] in the following definitions. Similar to the definition in finite-dimensional spaces, a collection of vectors $\{\varphi_k\}_{k=0}^\infty \subset L^2(\mathbb{R}^n)$ is called a *Schauder basis* if every $f \in L^2(\mathbb{R}^n)$ may be written uniquely as

$$\sum_{k=0}^\infty \alpha_k \varphi_k,$$

for $\alpha_k \in \mathbb{C}, k \geq 0$. Such a basis is called *unconditional* if for any sequence of coefficients $(\alpha_k)_{k=0}^\infty$, whenever $\sum_{k=0}^\infty \alpha_k\varphi_k$ converges (in the $\|\cdot\|_{L^2(\mathbb{R}^n)}$-norm), then so does $\sum_{k=0}^\infty |\alpha_k|\varphi_k$.

When $\{\varphi_k\}_{k=0}^\infty \subset L^2(\mathbb{R}^n)$ satisfies the bounds

$$A\|f\|^2_{L^2(\mathbb{R}^n)} \leq \sum_{k=0}^\infty |\langle f, \varphi_k\rangle_{L^2(\mathbb{R}^n)}|^2 \leq B\|f\|^2_{L^2(\mathbb{R}^n)}, \text{ for all } f \in L^2(\mathbb{R}^n),$$

for some $A, B > 0$, we call this a *frame*. If only the right-hand bound holds, this is called a *Bessel system*. When a frame is also a Schauder basis, it is called a *Riesz basis*, in which case it is unconditional [14]. When $A = B = 1$, a frame is called a *tight frame*. A special case of these are orthonormal bases, which are Schauder bases such that $\langle\varphi_j, \varphi_k\rangle_{L^2(\mathbb{R}^n)} = 0$ whenever $j \neq k$, and otherwise this equals 1. When $\{\varphi_k\}_{k=0}^\infty$ is a tight frame, and $\|\varphi_k\|_{L^2(\mathbb{R}^n)} = 1$ for all $k \geq 0$, it is an orthonormal basis [14, Prop. 3.2.1]. One of the major differences between frames and bases is that frames are redundant, so the representation of a function as $\sum_{k=0}^\infty \alpha_k\varphi_k$ may not be unique. This (potential) redundancy actually makes them preferred in certain applications (see [19, 51] and references within).

The positivity of the lower frame bound means that the *frame operator* $F : L^2(\mathbb{R}^n) \to \ell^2(\mathbb{N})$ given by $F(f) = \{\langle f, \varphi_k\rangle_{L^2(\mathbb{R}^n)}\}_{k=0}^\infty$ is invertible [14, Lem. 3.2.2], and if we denote

$\tilde{\varphi}_k = (F^*F)^{-1}\varphi_k$, where F^* is the adjoint of F, then $\{\tilde{\varphi}_k\}_{k=0}^{\infty}$ also forms a frame with bounds B^{-1} and A^{-1}. Defining $\tilde{F}$ analogously to F, we get the relations $\tilde{F}^*F = \mathrm{Id} = F^*\tilde{F}$ [14, Prop 3.2.3], leading us to call $\{\tilde{\varphi}_k\}$ the *dual frame*. This leads to the relations

$$f = \sum_{k=0}^{\infty} \langle f, \varphi_k\rangle_{L^2(\mathbb{R}^n)}\tilde{\varphi}_k = \sum_{k=0}^{\infty} \langle f, \tilde{\varphi}_k\rangle_{L^2(\mathbb{R}^n)}\varphi_k,$$

which are one reason that frames are nice systems for computation. In particular, tight frames are their own dual, which is particularly convenient. When both $\{\varphi_k\}_{k=0}^{\infty}, \{\tilde{\varphi}_k\}_{k=0}^{\infty}$ are Riesz bases for $L^2(\mathbb{R}^n)$, these are called *biorthogonal bases*, and similar to the orthogonal case, they must satisfy the relations $\langle \varphi_j, \tilde{\varphi}_k\rangle_{L^2(\mathbb{R}^n)} = 0$ if $j \neq k$, and 1 when $j = k$.

Example 1.1 (Special Systems in $\ell^2(\mathbb{N})$)**.** Consider $\{e_n\}_{n=0}^{\infty} \subset \ell^2(\mathbb{N})$, where e_n is the sequence with all entries equal to 0, except the nth entry, which is equal to 1. This is an orthonormal basis, since each element $x \in \ell^2(\mathbb{N})$ may be written uniquely as $\sum_{n=0}^{\infty} x_n e_n$, and $\langle e_j, e_k\rangle = 0$ whenever $j \neq k$, and equals 1 when $j = k$.

The set $\{\frac{1}{\sqrt{2}}e_n\}_{n=0}^{\infty} \cup \{\frac{1}{\sqrt{2}}e'_n\}_{n=0}^{\infty}$, where $e'_n = e_n$ for all $n \geq 0$, is a tight frame, but not a Schauder basis (and certainly not an orthonormal basis), since $\frac{1}{\sqrt{2}}e_0 - \frac{1}{\sqrt{2}}e'_0 = 0$ gives two different representations for 0 as $\sum \alpha_k \varphi_k$.

The set $\{(n+1)e_n\}_{n=0}^{\infty}$ is a Schauder basis, since $x = \sum_{n=0}^{\infty} \frac{x_n}{n+1}(n+1)e_n$, but not a Bessel system (and therefore not a frame or a Riesz basis), since $\sum_{n=0}^{\infty} |\langle e_k, (n+1)e_n\rangle_{\ell^2(\mathbb{N})}|^2 = (k+1)^2\|e_k\|^2_{\ell^2(\mathbb{N})}$, which means that there is no finite frame bound B. If we instead consider $\{e_n/(n+1)\}$, we have a Schauder basis which is a Bessel system with bound $B = 1$, but there is no positive lower frame bound A, so we do not have a frame, or a Riesz basis.

If we define $y_0 = e_0 + e_1$, $y_n = e_n$ for all $n \geq 1$, and $z_1 = e_1 - e_0$, $z_n = e_n$ for all $n \neq 1$, then $\{y_n\}, \{z_n\}$ are both Riesz bases, and the pair of them are biorthogonal bases. If we take $\{\frac{1}{\sqrt{2}}y_n\} \cup \{\frac{1}{\sqrt{2}}y'_n\}, \{\frac{1}{\sqrt{2}}z_n\} \cup \{\frac{1}{\sqrt{2}}z'_n\}$, with $y'_n = y_n, z'_n = z_n$ for all $n \geq 0$, then these are dual frames, but no longer Riesz bases. $\square$

Our goal in the current work will be to construct tight wavelet frames for $L^2(\mathbb{R}^n)$, which have even more structure. This additional structure is described in the following subsections.

1.2.4 Multiresolution Analyses and Wavelet Systems

So far, we have not reviewed any ideas specific to wavelets, but that changes here, with the fundamental concept of multiresolution analysis. At heart, a multiresolution analysis is just a collection of different subspaces of $L^2(\mathbb{R}^n)$, which we think of as representing a function at different resolutions. Let $\mathcal{M} \in M_n(\mathbb{Z})$ have eigenvalues outside the closed unit disk in $\mathbb{C}$, so that $\mathcal{M}$ is a dilation matrix, and let $\mathcal{Q} = |\det(\mathcal{M})|$. We will have more to say about such matrices in Section 1.3.1. A *multiresolution analysis (MRA)* is a collection of closed subspaces $V_j \subseteq L^2(\mathbb{R}^n), j \in \mathbb{Z}$ satisfying the following properties [14, 15]:

(i) $V_j \subset V_{j-1}$ for all $j \in \mathbb{Z}$.

(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^n)$.

(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

(iv) $f \in V_j$ if and only if $f(\mathcal{M}^j \cdot) \in V_0$.

(v) For all $f \in V_0$, $f(\cdot - k) \in V_0$ for all $k \in \mathbb{Z}^n$.

(vi) There is some $\phi \in L^2(\mathbb{R}^n)$ such that V_0 is the closed linear span of $\{\phi_{0,k} : k \in \mathbb{Z}^n\}$,

where for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, $\phi_{j,k}(\cdot) = \mathcal{Q}^{-j/2}\phi(\mathcal{M}^{-j} \cdot -k)$. The scaling here is chosen so that $\|\phi_{j,k}\|_{L^2(\mathbb{R}^n)} = \|\phi\|_{L^2(\mathbb{R}^n)}$ for all $j \in \mathbb{Z}, k \in \mathbb{Z}^n$. One can also see that the set of translations considered becomes finer for the spaces V_j at higher resolution: when $J > 0$, $\phi_{-J,k} \in V_{-J}$, where $\phi_{-J,k}(\cdot) = \mathcal{Q}^{J/2}\phi(\mathcal{M}^J(\cdot - \mathcal{M}^{-J}k))$. Then the shift $k \to k + e_m$ corresponds to a translation of $\mathcal{M}^{-J}e_m$ at this scale, which is necessarily small when J is large, since $\mathcal{M}$ is a dilation matrix. It is clear from the combination of (iv) and (vi) that for all $j \in \mathbb{Z}$, V_j is the closed linear span of $\{\phi_{j,k} : k \in \mathbb{Z}^n\}$. An MRA is called *local* if it is generated by a compactly supported function ϕ, which is known in any case as the *scaling* or *refinable function.* Now we can define wavelets, based on this notion of multiresolution analysis.

For a finite subset $\Psi \subset L^2(\mathbb{R}^n)$, we define the *wavelet system generated by the mother wavelets* Ψ by

$$\Lambda(\Psi) := \{\psi_{j,k} : \psi \in \Psi, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}, \tag{1.1}$$

where $\psi_{j,k}$ is defined as it was for ϕ in (vi). Given an MRA $(V_j)_{j\in\mathbb{Z}}$, we say that a wavelet system $\Lambda(\Psi)$ is *MRA-based* if $\Psi \subset V_{-1}$. Then an *MRA-based tight wavelet frame* is an MRA-based wavelet system $\Lambda(\Psi)$ which is also a tight frame for $L^2(\mathbb{R}^n)$. Since we only consider MRA-based tight wavelet frames, we will dispense with the "MRA-based" part of the description in the sequel.

It is well known that for a tight wavelet frame $\Lambda(\Psi)$, the number of mother wavelets is necessarily at least $\mathcal{Q}-1$, and this number is minimal when it is an orthonormal wavelet basis.

1.2.5 Connections between Wavelets and Trigonometric Polynomials

Looking more closely at the conditions for an MRA above, we see that $\phi \in V_0 \subset V_{-1}$, which means that

$$\phi = \mathcal{Q}^{-1/2} \sum_{k\in\mathbb{Z}^n} c_k \phi_{-1,k},$$

for some $c_k \in \mathbb{C}$, though in this work, we will be assuming that the $c_k \in \mathbb{R}$, for scaling functions. Moreover, when $\phi \in L^1(\mathbb{R}^n)$, normalizing $\int_{\mathbb{R}^n} \phi(x)\mathrm{d}x = 1$ implies that $\mathcal{Q}^{-1}\sum_{k\in\mathbb{Z}^n} c_k = 1$. Using the properties of the Fourier transform in Result 1.1(a) and (c), we see that

$$\begin{aligned}
\widehat{\phi_{-1,k}}(\omega) &= \mathcal{F}(\mathcal{Q}^{1/2}\phi(\mathcal{M}(x-\mathcal{M}^{-1}k)))(\omega) \\
&= \mathcal{Q}^{1/2}\mathcal{F}(\phi(\mathcal{M}\cdot))(\omega)e^{-i(\mathcal{M}^{-1}k)\cdot\omega} \\
&= \mathcal{Q}^{-1/2}\hat{\phi}(\mathcal{M}^{-T}\omega)e^{-ik\cdot(\mathcal{M}^{-T}\omega)}.
\end{aligned}$$

Then

$$\begin{aligned}
\hat{\phi}(\omega) &= \mathcal{Q}^{-1/2}\sum_{k\in\mathbb{Z}^n} c_k\widehat{\phi_{-1,k}}(\omega) \\
&= \mathcal{Q}^{-1}\sum_{k\in\mathbb{Z}^n} c_k\hat{\phi}(\mathcal{M}^{-T}\omega)e^{-ik\cdot(\mathcal{M}^{-T}\omega)} \\
&= \tau(\mathcal{M}^{-T}\omega)\hat{\phi}(\mathcal{M}^{-T}\omega), \qquad (1.2)
\end{aligned}$$

where $\tau(\omega) = \mathcal{Q}^{-1} \sum_{k \in \mathbb{Z}^n} c_k e^{-ik\cdot\omega}$ satisfies $\tau(0) = 1$. In our setting, we will suppose that τ is a trigonometric polynomial (this is actually the reason for scaling the Fourier transform as we have), so that only finitely many of the coefficients c_k are nonzero. We call τ a *lowpass mask*, about which we say more in Section 1.3.2. Similarly, for any $\psi \in \Psi \subset V_{-1}$, looking at the Fourier transform reveals that

$$\hat{\psi}(\omega) = q(\mathcal{M}^{-T}\omega)\hat{\phi}(\mathcal{M}^{-T}\omega) \tag{1.3}$$

for some *wavelet mask* $q(\omega) = \mathcal{Q}^{-1} \sum_{k \in \mathbb{Z}^n} d_k e^{-ik\cdot\omega}$. In our setting, we allow the d_k potentially to be in $\mathbb{C}$ rather than just $\mathbb{R}$, and we will consider trigonometric polynomial or rational trigonometric polynomial q. In the case that $q = r/s$ for some trigonometric polynomials r and s, where s is not a constant, then infinitely many of the d_k may be nonzero, and the convergence of this series becomes delicate. However, the formula (1.3) still makes sense in this setting, under some conditions on q, and we can find ψ by taking the inverse Fourier transform of the right hand side to get a function in $V_{-1} \subset L^2(\mathbb{R}^n)$, recalling that the V_j are closed subspaces. Some conditions under which this is possible are discussed in Chapter 5, Theorems 5.1 and 5.2, but especially in Section 5.1.2.

We may also reverse the direction of this argument, starting from a trigonometric polynomial lowpass mask, and arriving at an MRA. Iterating Equation (1.2), since τ is continuous with $\tau(0) = 1$, we would expect to obtain

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} \tau((\mathcal{M}^{-T})^j \omega).$$

When $\phi \in L^2(\mathbb{R}^n)$ satisfies this equation, we call it *the refinable function associated with* τ. It is a reasonable question to ask whether ϕ defined by this equation *is* a function, or an element of $L^2(\mathbb{R}^n)$, but the following lemma shows us that there is always some sense to be made of this equation, using the theory of distributions, as in [54]. This lemma is proved for a special case in [14, Ch. 6, Lem. 6.2.2].

Result 1.2 (Compactly Supported ϕ)**.** *Let* τ *be a trigonometric polynomial lowpass mask. Then* $\hat{\phi} = \prod_{j=1}^{\infty} \tau((\mathcal{M}^{-T})^j \cdot)$ *is an entire function of exponential type, and its distributional* *inverse Fourier transform ϕ is a compactly supported distribution.*

Obviously, we hope that ϕ is more than just a distribution, but a proper function belonging to $L^2(\mathbb{R}^n)$. The following lemma tells us that the sub-QMF condition on the lowpass mask guarantees that $\phi \in L^2(\mathbb{R}^n)$. We define the sub-QMF condition in Section 1.3.2, after we have introduced some more definitions, but it may be understood as controlling how large τ and its shifts by $2\pi\mathcal{M}^{-T}k$ get at any point, where $k \in \mathbb{Z}^n$. We introduce a generalization of this assumption in Chapter 5, for which the conclusion of this result also holds (see Remark 5.3).

Result 1.3 (Sub-QMF Trig. Polys yield L^2 Refin. Funcs)**.** *Suppose τ is a trigonometric polynomial that satisfies the sub-QMF condition. Then the refinable function ϕ corresponding to τ is a compactly supported function in $L^2(\mathbb{R}^n)$.*

In light of these results, we are free to pursue the construction of tight wavelet frames from the perspective of designing trigonometric polynomials satisfying certain conditions, which is the approach we will take in this work. Clearly, obtaining all of the properties of a multiresolution analysis will require some additional conditions on τ, and obtaining tight wavelet frames $\Lambda(\Psi)$ will require some additional conditions on the functions q defining the mother wavelets ψ as in Equation (1.3). We discuss some possible conditions for accomplishing this in Section 1.5, but first we give some more definitions.

1.3 Filters and Masks in the Context of Wavelets

In this section, we further discuss properties of trigonometric polynomials and rational trigonometric polynomials which will be used extensively throughout this work.

1.3.1 Dilation Matrices and Group Actions

We say that $\mathcal{M} \in M_n(\mathbb{Z})$ is a *dilation matrix* when the eigenvalues of $\mathcal{M}$ lie outside of the closed unit disk, so that $\sigma(\mathcal{M}) \subset \{z \in \mathbb{C} : |z| > 1\}$. We denote the unsigned determinant of the dilation matrix by $\mathcal{Q} = |\det(\mathcal{M})|$.

We recall that a finite group G is said to *act on a set* X when there is an associated permutation of X for each element of G, such that the identity element of G acts as the identity permutation, and $g_1 \circ (g_2 \circ x) = (g_1 g_2) \circ x$ for all $g_1, g_2 \in G$ and $x \in X$, where we denote the permutation of X associated with $g \in G$ by $g \circ x$ for all $x \in X$, and we denote the group multiplication by juxtaposition. Then the *orbit* of $x \in X$ is the set $G \circ x := \{y \in X : y = g \circ x \text{ for some } g \in G\}$.

There are two major group actions that we consider in the following. The first is that of the group $\mathcal{G} := (2\pi\mathcal{M}^{-T}\mathbb{Z}^n)/(2\pi\mathbb{Z}^n)$ acting on trigonometric polynomials and rational trigonometric polynomials. Let Γ^* be a set of distinct coset representatives for $\mathcal{G}$ including 0. Then for $\gamma \in \Gamma^*$ and a rational trigonometric polynomial f, we define $\gamma : f \mapsto f^\gamma := f(\cdot + \gamma)$. This definition is independent of the set of coset representatives for $\mathcal{G}$, since if $\gamma_1 \equiv \gamma_2 \pmod{2\pi\mathbb{Z}^n}$, there is some $k \in \mathbb{Z}^n$ such that $\gamma_1 = \gamma_2 + 2\pi k$, so $f^{\gamma_1} = f(\cdot + \gamma_1) = f(\cdot + \gamma_2 + 2\pi k) = f(\cdot + \gamma_2) = f^{\gamma_2}$, by 2π-periodicity. The other group action that we consider will be described in Section 3.3.1, since we only use it in that chapter.

Definition 1.1. We say that a rational trigonometric polynomial f is $\mathcal{G}$-invariant if for all $\gamma \in \Gamma^*$, $f^\gamma = f$.

We say that $H(\omega)$ is a $\mathcal{G}$-vector for the rational trigonometric polynomial τ if $H(\omega) = [\tau^\gamma(\omega)]_{\gamma \in \Gamma^*}$. We also call a vector a $\mathcal{G}$-vector if it is of this form for some rational trigonometric polynomial. □

The following simple lemma is well-known, but we include a proof for completeness.

Lemma 1.1 ($\mathcal{G}$-invariance)**.** *Let f be a rational trigonometric polynomial. Then f is $\mathcal{G}$-invariant if and only if there is a rational trigonometric polynomial g such that $f = g(\mathcal{M}^T \cdot)$.*

Proof. If g is a rational trigonometric polynomial, then $g(\mathcal{M}^T \cdot)$ is $\mathcal{G}$-invariant, since

$$(g(\mathcal{M}^T \cdot))^\gamma(\omega) = g(\mathcal{M}^T(\omega + \gamma)) = g(\mathcal{M}^T \omega),$$

because $\mathcal{M}^T\gamma \in 2\pi\mathbb{Z}^n$. On the other hand, if $f = p/q$ is $\mathcal{G}$-invariant, then

$$f = \mathcal{Q}^{-1} \sum_{\gamma \in \Gamma^*} \frac{p^\gamma}{q^\gamma} = \mathcal{Q}^{-1} \left(\prod_{\gamma \in \Gamma^*} \frac{1}{q^\gamma} \right) \sum_{\gamma \in \Gamma^*} p^\gamma \prod_{\gamma' \neq \gamma} q^{\gamma'},$$

where $q' = \prod_\gamma (q^\gamma)^{-1}$ is clearly $\mathcal{G}$-invariant, and since $f = p'/q'$ means that $p' = fq'$, p' is $\mathcal{G}$-invariant, so it suffices to prove the result for trigonometric polynomials. When $f(\omega) = \sum_{k\in\mathbb{Z}^n} f_k e^{-ik\cdot\omega}$, then by $\mathcal{G}$-invariance,

$$\begin{aligned}
f(\omega) &= \mathcal{Q}^{-1} \sum_{\gamma\in\Gamma^*} f^\gamma(\omega) \\
&= \mathcal{Q}^{-1} \sum_{\gamma\in\Gamma^*} \sum_{k\in\mathbb{Z}^n} f_k e^{-ik\cdot(\omega+\gamma)} \\
&= \sum_{k\in\mathbb{Z}^n} f_{\mathcal{M}k} e^{-ik\cdot\mathcal{M}^T\omega},
\end{aligned}$$

using the relation for $k \in \mathbb{Z}^n$ that $\sum_{\gamma\in\Gamma^*} e^{ik\cdot\gamma} = \mathcal{Q}$ when $k \equiv 0 \,(\mathrm{mod}\, \mathcal{M}\mathbb{Z}^n)$, and equals 0 otherwise [43, Ch. XVIII]. Since this last expression is clearly a trigonometric polynomial in $\mathcal{M}^T\omega$, we are done. □

After introducing the polyphase representation in Section 1.3.3, we will see that in the case f is $\mathcal{G}$-invariant, it is equal to $\mathcal{Q}^{-1/2} f_0(\mathcal{M}^T\omega)$, where f_0 is the polyphase component corresponding to the coset $0 + \mathcal{M}\mathbb{Z}^n$ in $\mathbb{Z}^n/\mathcal{M}\mathbb{Z}^n$.

While some of the results in this work will hold for any dilation matrix, in many cases we will consider scalar dilation, where $\mathcal{M} = pI$ for some prime number p. In this setting, Γ^* will usually taken to be $(2\pi/p)\{0, 1, \ldots, p-1\}^n$.

1.3.2 Filters and Masks

We say that τ, a trigonometric polynomial, is a *mask* associated with the *filter* $h : \mathbb{Z}^n \to \mathbb{R}$ (or $\mathbb{C}$), which is only nonzero at finitely many points, if it is the Fourier transform of h; i.e., $\tau(\omega) = \mathcal{Q}^{-1} \sum_{k\in\mathbb{Z}^n} h(k) e^{-ik\cdot\omega}$, with $\omega \in \mathbb{T}^n := [-\pi, \pi]^n$, where n is the spatial dimension. This normalization is clearly chosen with a certain dilation matrix in mind, and in practice, this will always be clear. We will reserve R and H as the symbols for mask and filter, respectively, in the case that $n = 1$, to differentiate between the input and output masks and filters for the coset sum and prime coset sum operators (see Sections 1.4.1 and 1.4.2). When a statement is made for any dimension $n \geq 1$ (and especially if it is made for $n \geq 2$), we will use τ and h.

We say that a filter $h : \mathbb{Z}^n \to \mathbb{R}$ is *lowpass*, or *refinement*, if $\sum_{k\in\mathbb{Z}^n} h(k) = \mathcal{Q}$, and if this sum is instead equal to 0, we say that the filter is *highpass*, or *wavelet*.

An important quantity associated with a lowpass mask is its *accuracy number*, which is the minimum order of the roots this lowpass mask has at the nonzero elements of Γ^*. When a lowpass mask does not have a root at one of the nonzero elements of Γ^*, it is said to have accuracy number zero, though we will typically consider lowpass masks with positive accuracy number. The other quantity of interest for lowpass masks is the *flatness number*, which is the order of the root of $1-\tau$ at $\omega = 0$. The order of the root that a highpass mask has at 0 is called its number of *vanishing moments*, which is necessarily at least one, by definition. In an (MRA-based) orthonormal wavelet system, the accuracy number of the lowpass mask τ determines the number of vanishing moments of the associated highpass masks. In any wavelet system, if all of the highpass masks have $l \geq 1$ vanishing moments, all polynomials of degree at most l lie in the subspace of translations of the refinable function. As l increases, this results in faster convergence in the approximation of $L_2(\mathbb{R}^n)$ functions by the wavelet system [57].

Another important property of a lowpass mask is the *interpolatory property.* For $n \geq 1$, if a lowpass mask τ satisfies:

$$\sum_{\gamma\in\Gamma^*} \tau^\gamma \equiv 1, \tag{1.4}$$

we say that τ has the interpolatory property, or is *interpolatory.* A wealth of additional information about wavelet transforms with interpolatory masks may be found in [20]. One special property of interpolatory lowpass masks is that their flatness number is always at least their accuracy number. Since

$$1 - \tau = \sum_{\gamma\in\Gamma^*\backslash\{0\}} \tau^\gamma,$$

this relation is clearly seen to be true.

One last property that we define for lowpass masks is called the sub-QMF condition, which we already saw is sufficient for the associated refinable function to belong to $L^2(\mathbb{R}^n)$, in Result 1.3. This condition will appear several times in the sequel. We say that a lowpass mask τ satisfies the sub-QMF condition when

$$f(\tau;\omega) := 1 - \sum_{\gamma\in\Gamma^*} |\tau^\gamma(\omega)|^2 \geq 0, \quad \forall \omega \in \mathbb{T}^n. \tag{1.5}$$

Note that this implies that τ has positive accuracy number, since $\sum_{\gamma\in\Gamma^*\setminus\{0\}} |\tau(\gamma)|^2 \leq 1 - |\tau(0)|^2 = 0$, using the lowpass condition $\tau(0) = 1$.

The QMF condition (where $f(\tau;\cdot) \equiv 0$) has been studied extensively in the wavelet literature [56, 58], especially in the context of orthonormal wavelet basis construction, where it is necessary [1, 60].

For convenience, we will also sometimes say that a filter has a certain accuracy number or some other property of masks, and this should be understood to mean that the mask associated with this filter has the specified accuracy number or property. Similarly, if we say that a mask has a property usually associated with filters, this should be interpreted to mean that the filter associated with this mask has the stated property.

1.3.3 Polyphase Representation

In the context of wavelet construction, it is frequently convenient to have several different ways of representing the same trigonometric polynomial: besides the mask and filter, we have also considered the group action of $\mathcal{G}$ on trigonometric polynomials. The *polyphase representation* is yet another way of representing a trigonometric polynomial or rational trigonometric polynomial. Let Γ be a set of distinct coset representatives for $\mathbb{Z}^n/\mathcal{M}\mathbb{Z}^n$ containing 0. Then we have the following definition.

Definition 1.2. For f a rational trigonometric polynomial, we define its polyphase components f_ν, $\nu \in \Gamma$, by

$$f_\nu(\mathcal{M}^T\omega) = \mathcal{Q}^{-1/2} \sum_{\gamma\in\Gamma^*} f^\gamma(\omega) e^{-i(\omega+\gamma)\cdot\nu}. \tag{1.6}$$

□

We will also use the following dual relation, which is easy to show using the definition above:

$$f(\omega) = \mathcal{Q}^{-1/2} \sum_{\nu \in \Gamma} f_\nu(\mathcal{M}^T \omega) e^{i\omega \cdot \nu}. \tag{1.7}$$

One way of thinking about these transformations is to consider the $\mathcal{G}$-vector generated by f, $[f^\gamma]_{\gamma \in \Gamma^*}$, and the *polyphase vector generated by* f, $[f_\nu]_{\nu \in \Gamma}$, for some orderings of the sets Γ^*, Γ. Let $X(\omega)$ be the Fourier transform matrix $\mathcal{Q}^{-1/2}[e^{i(\omega+\gamma)\cdot\nu}]_{\gamma \in \Gamma^*, \nu \in \Gamma}$. We have the following relationship between the $\mathcal{G}$-vector and polyphase vector for f, which is just a different way of writing the equations above:

$$[f_\nu(\mathcal{M}^T \cdot)]_{\nu \in \Gamma} = X(\omega)^* [f^\gamma]_{\gamma \in \Gamma^*}, \quad [f^\gamma]_{\gamma \in \Gamma^*} = X(\omega)[f_\nu(\mathcal{M}^T \cdot)]_{\nu \in \Gamma}. \tag{1.8}$$

When f is a trigonometric polynomial, we may compute the filter coefficients of f_ν from Equation (1.6), and this was done for the special case of $\nu = 0$ in the proof of Lemma 1.1. When f is associated with the filter h, this gives the equation

$$f_\nu(\omega) := \mathcal{Q}^{-1/2} \sum_{k \in \mathbb{Z}^n} h(\mathcal{M}k - \nu) e^{-ik\cdot\omega}. \tag{1.9}$$

When τ is an interpolatory lowpass mask, from Equations (1.6) and (1.4), we get the equality

$$\tau_0(\mathcal{M}^T \omega) = \mathcal{Q}^{-1/2} \sum_{\gamma \in \Gamma^*} \tau^\gamma(\omega) = \mathcal{Q}^{-1/2}, \text{ for all } \omega \in \mathbb{T}^n. \tag{1.10}$$

1.3.4 Other Notation

Here and below, we use $\geq_{\text{lex}}$ to denote the lexicographic order on $\mathbb{Z}^n$. That is, for $x, y \in \mathbb{Z}^n$, $x \geq_{\text{lex}} y$ if $x = y$, or if $x \neq y$, and in the first position (reading left to right) such that $x_i \neq y_i$, $x_i > y_i$. We will write $>_{\text{lex}}$ to denote the case when equality is excluded. When this ordering is used, the choice of order is rarely important, under some mild assumptions, but we choose lexicographic for convenience.

We will write $\alpha \geq_{\text{e}} \beta$ to say that $\alpha_i \geq \beta_i$ for all $1 \leq i \leq n$ (the "entrywise" comparison). When $\alpha \geq_{\text{e}} 0$, we write $|\alpha| = \sum_{j=1}^n \alpha_j$, and for $\alpha \geq_{\text{e}} \beta \geq_{\text{e}} 0$, we let $\binom{\alpha}{\beta} = \prod_{j=1}^n \binom{\alpha_j}{\beta_j}$. For

$a, b \in \mathbb{C}^n$ and $\alpha \geq_{\mathrm{e}} 0$, this gives the formula

$$(a+b)^\alpha = \sum_{0 \leq_{\mathrm{e}} k \leq_{\mathrm{e}} \alpha} \binom{\alpha}{k} a^k b^{\alpha-k}, \tag{1.11}$$

which is the result of applying the binomial theorem to each factor $(a_j+b_j)^{\alpha_j}$ and expanding.

We use the multiindex notation as in [24], which we very briefly review here. For a multiindex $\alpha \in \mathbb{Z}^n$, $\alpha \geq_{\mathrm{e}} 0$, we denote by D^α the partial derivative operator

$$\frac{\partial^{|\alpha|}}{\prod_{j=1}^n \partial \omega_j^{\alpha_j}}.$$

This yields the formula $D^\alpha \exp(ik \cdot \omega) = i^{|\alpha|} k^\alpha \exp(ik \cdot \omega)$, where $k^\alpha = \prod_{j=1}^n k_j^{\alpha_j}$. We will also use Leibniz's formula

$$D^\alpha(uv) = \sum_{0 \leq_{\mathrm{e}} \beta \leq_{\mathrm{e}} \alpha} \binom{\alpha}{\beta} D^\beta u \, D^{\alpha-\beta} v, \tag{1.12}$$

where the sum is over all multiindices β satisfying $0 \leq_{\mathrm{e}} \beta \leq_{\mathrm{e}} \alpha$.

1.4 Coset Sum and Prime Coset Sum Methods

In this section, we review the coset sum [38] and prime coset sum [39] operators, which lift a univariate lowpass mask with prime dilation to construct a nonseparable multivariate lowpass mask with this same prime dilation, and preserve some of the other properties of the input mask. We will use these two operators in Chapters 2 and 3 to construct nonseparable multivariate tight wavelet frames with prime dilation.

1.4.1 Coset Sum

Let us recall the definition of the coset sum (CS) operator [38]. We assume that the univariate dilation factor is 2, and that the multivariate dilation matrix is $2I$, where I is the $n \times n$ identity matrix, for some $n \geq 2$. For definiteness, we take $\Gamma^* = \{0, \pi\}^n$ and $\Gamma = \{0, 1\}^n$, and we let $\Gamma' = \Gamma \setminus \{0\}$.

Let $R(\omega) = \frac{1}{2} \sum_{k \in \mathbb{Z}} H(k) e^{-ik\omega}$, be the mask associated with the univariate lowpass filter

H, so that $R(0) = 1$. Then the output of the CS operator into n dimensions is given by

$$\tau(\omega) := \mathcal{C}_n[R](\omega) := \frac{1}{2^n}\left(2 - 2^n + 2\sum_{\nu\in\Gamma'} R(\nu\cdot\omega)\right), \quad \omega \in \mathbb{T}^n. \tag{1.13}$$

As usual, the associated filter h is defined via $\tau(\omega) =: 2^{-n}\sum_{k\in\mathbb{Z}^n} h(k)e^{-ik\cdot\omega}$. CS-generated lowpass filters in n dimensions have a star shape, as can be seen in the case of two dimensions in Figures 2.1(a) and 2.2, as well as [38, Figures 4,5]. This can be seen from the definition, since

$$\tau(\omega) = \frac{1}{2^n}\left(h(0) + \sum_{\nu\in\Gamma'}\sum_{k\neq 0} H(k)e^{-ik(\nu\cdot\omega)}\right),$$

so h only takes nonzero values on $\bigcup_{\nu\in\Gamma'} \operatorname{span}(\nu)$.

Result 1.4 (Properties of CS [38])**.** *Let R be a univariate lowpass mask, and let $\tau = C_n[R]$ be the output of the coset sum in n dimensions. The following properties hold:*

(a) R is interpolatory if and only if τ is interpolatory.

(b) If R has accuracy number a and flatness number b, then τ has accuracy number at least $\min\{a, b\}$.

(c) Suppose that R is interpolatory. Then τ and R have the same accuracy number.

1.4.2 Prime Coset Sum

The prime coset sum (PCS) method generalizes CS to any prime dilation factor p. For a specified dimension $n \geq 2$, given the input lowpass mask R and a set Γ of distinct coset representatives of $\mathbb{Z}^n/p\mathbb{Z}^n$ containing 0, the PCS lowpass mask is defined for all $\omega \in \mathbb{T}^n$ as

$$\tau(\omega) := \mathcal{C}_{n,p}[R](\omega) := \frac{1}{(p-1)p^{n-1}}\left(1 - p^{n-1} + \sum_{\nu\in\Gamma'} R(\omega\cdot\nu)\right). \tag{1.14}$$

Note that this definition depends on the choice of Γ, and some of the properties of τ will depend on this choice. Here, the dilation matrix is pI, so the filter is defined via the relation $\tau(\omega) = p^{-n}\sum_{k\in\mathbb{Z}^n} h(k)e^{-ik\cdot\omega}$. We review some properties of PCS below, and will prove more in Chapter 3.

Result 1.5 (Properties of PCS [39]). *Let R be a univariate lowpass mask with prime dilation p, and let $\tau = \mathcal{C}_{n,p}[R]$ be the output of the prime coset sum in n dimensions. Then the following properties hold:*

(i) If R is interpolatory, then τ is interpolatory.

(ii) If R has accuracy number a and flatness number b, then τ has accuracy number at least $\min\{a, b\}$.

(iii) If $R(\omega) = R(-\omega)$, then $\tau(\omega) = \tau(-\omega)$.

1.5 Extension Principles

We have seen that under certain conditions on a lowpass mask τ, the associated refinable function belongs to $L^2(\mathbb{R}^n)$, allowing us to define an MRA. Then to get an MRA-based tight wavelet frame, we need to find a finite collection of functions $\Psi \subset V_{-1}$ such that $\Lambda(\Psi)$ is a tight frame. In light of Equation 1.3, we will try to accomplish this by constructing rational trigonometric polynomials $q_j, 1 \leq j \leq r$, such that the associated mother wavelets $\psi^{(j)}$ generate a tight wavelet frame. It turns out that because of the structure of the wavelet system $\Lambda(\Psi)$, we can turn the analytic conditions for this set being a tight wavelet frame into (mostly) algebraic conditions on the rational trigonometric polynomials q_j. Two of the major results in this area are referred to as "extension principles," and give sets of conditions under which this idea goes through. We will prove some variations of these results in Chapter 5.

1.5.1 Unitary Extension Principle

The first extension principle we consider is the *unitary extension principle* (UEP), which provides a systematic way to construct a tight wavelet frame [29, 49]. It consists of a set of conditions on a collection of trigonometric polynomials τ, q_j, where $1 \leq j \leq r$, such that $\Lambda(\{\psi^{(1)}, \ldots, \psi^{(r)}\})$ (see Equation (1.1)) is a tight frame. The following version of the theorem comes from [30]. For a version allowing rational trigonometric polynomial masks, see Corollary 5.1.

Result 1.6 (UEP)**.** *Let τ be a trigonometric polynomial with $\tau(0) = 1$, and let ϕ be defined by $\hat{\phi}(\omega) := \prod_{j=1}^{\infty} \tau((\mathcal{M}^{-T})^j \omega)$ for $\omega \in \mathbb{R}^n$. If q_j, $1 \leq j \leq r$, are trigonometric polynomials such that for all $\omega \in \mathbb{T}^n$ and $\gamma \in \Gamma^*$:*

$$\tau(\omega)\overline{\tau(\omega+\gamma)} + \sum_{j=1}^{r} q_j(\omega)\overline{q_j(\omega+\gamma)} = \begin{cases} 1, & \gamma = 0 \\ 0, & \textit{otherwise}, \end{cases} \tag{1.15}$$

then $\Lambda(\{\psi^{(1)}, \ldots, \psi^{(r)}\})$ is a tight wavelet frame for $L_2(\mathbb{R}^n)$.

When a set of wavelet masks q_j, $1 \leq j \leq r$ satisfy the UEP conditions as above with some lowpass mask τ, we will call the set $\{\tau, q_1, \ldots, q_r\}$ a *tight wavelet filter bank*. We will construct tight wavelet frames with the UEP in Chapters 2 and 3.

One of the statements being asserted in this result is that the refinable function ϕ belongs to $L^2(\mathbb{R}^n)$. This actually follows from Result 1.3, since the UEP conditions (1.15) imply that τ satisfies the sub-QMF condition. The following argument from [11] demonstrates this: let $T(\omega)$ be a $\mathcal{G}$-vector for τ, for some ordering of Γ^*, and let $Q(\omega) = [q_j^{\gamma}]_{\gamma \in \Gamma^*, j \in \{1,\ldots,r\}}$. Then the conditions (1.15) are equivalent to $I - T(\omega)T(\omega)^* = Q(\omega)Q(\omega)^*$, and taking determinants on both sides, $f(\tau;\omega) = \det(Q(\omega)Q(\omega)^*) \geq 0$ for all $\omega \in \mathbb{T}^n$, since $Q(\omega)Q(\omega)^*$ is always positive semidefinite. We will see that this relationship has even stronger implications in Section 1.7.1.

1.5.2 Oblique Extension Principle

One issue with constructions using the UEP is that the number of vanishing moments of the constructed wavelets may lag far behind the accuracy number of the lowpass mask. In other words, the MRA has the potential to give rise to a tight wavelet frame with very fast convergence properties, but the constructed wavelet frames do not exhibit this behavior. One method for correcting this is by introducing a *vanishing moment recovery function*, and finding wavelet masks with more vanishing moments satisfying a modified set of the conditions (1.15). The extension principle associated with this approach is called the *oblique extension principle* (UEP), the statement of which is given below, from [15].

This uses the notation $\langle u, v\rangle_w = wu_0\overline{v_0} + \sum_{i=1}^{r} u_i\overline{v_i}$, as well as $\sigma(\tau) := \{\omega \in \mathbb{T}^n :$

$\hat{\phi}(\omega + 2\pi k) \neq 0,$ for some $k \in \mathbb{Z}^n\}$, where ϕ is the refinable function associated with τ. We have adapted the notation of this theorem to our setting.

Result 1.7 (OEP). *Let $\tau, q_1, \ldots, q_r$ be 2π-periodic functions, and let $\boldsymbol{\tau} = (\tau, q_1, \ldots, q_r)$ be the combined MRA mask. Suppose that*

(a) Each mask τ, q_j in the combined MRA mask $\boldsymbol{\tau}$ belongs to $L^\infty(\mathbb{T}^n)$.

(b) The refinable function ϕ satisfies $\lim_{\omega \to 0} \hat{\phi}(\omega) = 1$.

(c) The function $[\hat{\phi}, \hat{\phi}] := \sum_{k \in \mathbb{Z}^n} |\hat{\phi}(\cdot + 2\pi k)|^2 \in L^\infty(\mathbb{T}^n)$.

Suppose that S is a 2π-periodic function that satisfies the following:

(i) $S \in L^\infty(\mathbb{T}^n)$ is nonnegative, continuous at the origin, and $S(0) = 1$.

(ii) If $\omega \in \sigma(\tau)$, and if $\gamma \in \Gamma^*$ is such that $\omega + \gamma \in \sigma(\tau)$, then

$$\langle \boldsymbol{\tau}(\omega), \boldsymbol{\tau}(\omega + \gamma) \rangle_{S(\mathcal{M}^T\omega)} = S(\omega)\delta(\gamma), \tag{1.16}$$

where $\delta(\gamma) \in \{0, 1\}$ for all $\gamma \in \Gamma^*$, and only $\delta(0) = 1$.

Then the wavelet system defined by $\boldsymbol{\tau}$ is a tight wavelet frame for $L^2(\mathbb{R}^n)$.

In this work, we will always be considering τ as a trigonometric polynomial lowpass mask. In this case, $\sigma(\tau) = \mathbb{T}^n$ [15]. We also note that when ϕ satisfies condition (c), it necessarily belongs to $L^2(\mathbb{R}^n)$, since if $\|[\hat{\phi}, \hat{\phi}]\|_{L^\infty(\mathbb{T}^n)} \leq C$, then

$$\|\phi\|^2_{L^2(\mathbb{R}^n)} = \|\hat{\phi}\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\hat{\phi}(\omega)|^2 \mathrm{d}\omega = \int_{[-\pi,\pi]^n} [\hat{\phi}, \hat{\phi}](\omega) \mathrm{d}\omega \leq C(2\pi)^n.$$

The function S is called the *vanishing moment recovery (vmr) function*, since the additional flexibility it brings may be used to construct highpass masks with better vanishing moments than in the unitary extension principle setting, which forces $S \equiv 1$.

We will construct tight wavelet frames with the OEP in Chapter 5.

1.6 Sums of Squares Representations

In this section, we describe some spaces of polynomials and rational polynomials, as well as some spaces of trigonometric polynomials. We then introduce the notion of sums of squares (sos) representations and sums of rational squares (sors) representations for a non-negative polynomial or trigonometric polynomial. After introducing some of the theory of formally real fields, which we will need in Chapter 4, we give some results about when so(r)s representations exist. Throughout this work, we will use the theory of sums of squares representations to find collections of trigonometric polynomials satisfying the conditions of the extension principles introduced in the last section. In Chapter 4, we prove some new results about the existence of so(r)s representations in a few contexts, which will be useful for the OEP-based constructions we consider in Chapter 5.

1.6.1 Definitions

We denote by $\mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, \ldots, z_n]$ the ring of polynomials in the variables $z_1, \ldots, z_n$ with coefficients in $\mathbb{C}$, and similarly for $\mathbb{R}[\mathbf{z}]$. We denote by $\mathbb{C}(\mathbf{z}) = \mathbb{C}(z_1, \ldots, z_n)$, or the field of fractions of $\mathbb{C}[\mathbf{z}]$, which is the space of rational polynomials $f = p/q$ where $p, q \in \mathbb{C}[\mathbf{z}]$ and $q \neq 0$, and similarly for $\mathbb{R}(\mathbf{z})$. We will also refer to Laurent polynomials, which are the elements of $\mathbb{C}(\mathbf{z})$ of the form $z_1^{k_1} \cdots z_n^{k_n} p$, where $p \in \mathbb{C}[\mathbf{z}]$ and $k_1, \ldots, k_n \in \mathbb{Z}$. We will denote the set of Laurent polynomials as $\mathbb{C}[\mathbf{z}^{\pm 1}]$, or $\mathbb{R}[\mathbf{z}^{\pm 1}]$ when the coefficient field is $\mathbb{R}$.

Remark 1.1 (Spaces of Rational Functions)**.** For the sake of clarity, we point out that the space of rational trigonometric polynomials is equivalent to the restriction of functions in $\mathbb{C}(\mathbf{z})$ to evaluation at $z \in (\partial\mathbb{D})^n$, where $\mathbb{D}$ is the complex unit disk. That is, $(\partial\mathbb{D})^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| = 1 \text{ for all } j\}$. This is also equivalent to the same restriction for functions in the field of quotients of Laurent polynomials with complex coefficients. In either case, the trigonometric polynomial S associated with the polynomial or Laurent polynomial f is defined by $S(\omega) = f(e^{i\omega_1}, \ldots, e^{i\omega_n})$ for all $\omega \in \mathbb{T}^n := [-\pi, \pi]^n$, and similarly for a rational trigonometric polynomial S. In particular, a trigonometric polynomial is any function which is a sum of the form $\sum c_k e^{-ik\cdot\omega}$, where k ranges over some finite subset of $\mathbb{Z}^n$, and $k \cdot \omega$ denotes the ordinary Euclidean inner product in $\mathbb{R}^n$.

Recall that irreducible polynomials (see, e.g., [43]) $p, q \in \mathbb{C}[\mathbf{z}]$ are called *associated* when $p = wq$, for $w \in \mathbb{C} \setminus \{0\}$. For any $p \in \mathbb{C}[\mathbf{z}] \setminus \{0\}$, $p = up_1 \cdots p_r$, where $u \in \mathbb{C} \setminus \{0\}$, and $p_1, \ldots, p_r$ are irreducible polynomials. Moreover, this representation is unique in the sense that if $p = wq_1 \cdots q_s$, with $w \in \mathbb{C}$ and $q_1, \ldots, q_s$ irreducible polynomials, then $s = r$, and for all $1 \le i \le r$, q_i is associated with some $p_j, 1 \le j \le r$. This means that if $f \in \mathbb{C}(\mathbf{z}) \setminus \{0\}$, it has a representation as $\frac{up_1 \cdots p_r}{q_1 \cdots q_s}$, where $u \in \mathbb{C} \setminus \{0\}$, and $p_i, 1 \le i \le r, q_j, 1 \le j \le s$ are irreducible polynomials, such that for all i and j, it is not the case that p_i and q_j are associated. Let p be the numerator of the previous expression, and let q be the denominator. If we have any representation of f as v/t, where $v, t \in \mathbb{C}[\mathbf{z}]$, then since $p/q = v/t$, $pt = vq$, and since every p_i must divide vq, but p_i is not associated with any q_j, $p_i | v$ for all $1 \le i \le r$. Similarly, $q_j | t$ for all $1 \le j \le s$, and any further irreducible factors of either side of this equation are common to both v and t (up to association).

Putting this differently, let $f = p/q$, such that p and q are as in the previous paragraph. Given any representation of $f = v/t$, there exists some polynomial m such that $v = pm$, $t = qm$. We will say that $f = p/q$ is in *lowest terms* when this holds.

When we discuss the set of points where $f \in \mathbb{C}(\mathbf{z})$ is defined, we mean the set of points $z \in \mathbb{C}^n$ for which $q(z) \neq 0$, where $f = p/q$ is in lowest terms (noting that this set is invariant under association of the irreducible factors of q). $\square$

Definition 1.3. We say that a polynomial or Laurent polynomial $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ has a *sum of hermitian squares representation (sos representation) on $\Omega \subseteq \mathbb{C}^n$ with $\{g_j\}_{j=1}^J \subset \mathbb{C}[\mathbf{z}]$, $J < +\infty$,* when

$$f(z) = \sum_{j=1}^{J} |g_j(z)|^2 \quad \forall z \in \Omega. \tag{1.17}$$

We may abbreviate this when the set Ω is clear (we will typically consider $\mathbb{R}^n$ or $(\partial\mathbb{D})^n$), or just say that f has an sos representation or is an sos on Ω if there is some finite collection of $g_j \in \mathbb{C}[\mathbf{z}]$ for which this equation holds for all $z \in \Omega$.

Similarly, we say that $f \in \mathbb{C}(\mathbf{z})$ has a *sum of rational hermitian squares representation (sors representation) on $\Omega \subseteq \mathbb{C}^n$ with $\{g_j\}_{j=1}^J \subset \mathbb{C}(\mathbf{z})$*, if Equation (1.17) holds for all $z \in \Omega$ at which $f(z)$ is defined.

Using the identifications described in Remark 1.1, we may also apply these definitions in the case that f and the g_j are trigonometric polynomials, or rational trigonometric polynomials. In this case, the natural domain to consider will be $\Omega = \mathbb{T}^n$. $\square$

Recalling the definition of the group action of $\mathcal{G}$ on rational trigonometric polynomials from Section 1.3.1, we can also define $\mathcal{G}$-invariant so(r)s representations.

Definition 1.4. Let f be a rational trigonometric polynomial with an sos or sors of functions $g_j, 1 \leq j \leq J$. We say that f has a *$\mathcal{G}$-invariant so(r)s* if its so(r)s representation has the property that $g_j^\gamma = g_j$ for all $\gamma \in \mathcal{G}$, for all $1 \leq j \leq J$. $\square$

We now prove a few lemmas regarding $\mathcal{G}$-invariance and sums of squares representations. Similar results appear in [11, Lemma 2.1] under more restrictive assumptions. Part (a) of this lemma might be viewed as asserting that the Fourier transform matrix $X(\omega)$ in Section 1.3.3 is unitary.

Lemma 1.2 ($\mathcal{G}$-invariance)**.** *Let f be a rational trigonometric polynomial.*

(a) $\sum_{\gamma\in\Gamma^*} |f^\gamma|^2 = \sum_{\nu\in\Gamma} |f_\nu(\mathcal{M}^T\cdot)|^2$.

(b) If f is $\mathcal{G}$-invariant, then it is an so(r)s if and only if it is a $\mathcal{G}$-invariant so(r)s.

Proof. (a) We use Equation (1.7) to compute, for $\omega \in \mathbb{T}^n$ such that the left hand side is defined:

$$\sum_{\gamma\in\Gamma^*} |f^\gamma(\omega)|^2 = \mathcal{Q}^{-1} \sum_{\gamma\in\Gamma^*} \sum_{\nu,\nu'\in\Gamma} f_\nu(\mathcal{M}^T\omega)\overline{f_{\nu'}(\mathcal{M}^T\omega)} e^{i(\omega+\gamma)\cdot(\nu-\nu')} = \sum_{\nu\in\Gamma} |f_\nu(\mathcal{M}^T\omega)|^2,$$

using the fact that for $k \in \mathbb{Z}^n$, $\sum_{\gamma\in\Gamma^*} e^{ik\cdot\gamma}$ is equal to $\mathcal{Q}$ when $k \equiv 0 \,(\mathrm{mod}\, \mathcal{M}\mathbb{Z}^n)$ and is 0 otherwise [43, Ch. XVIII].

(b) The converse is obvious, so if $f = \sum_{j=1}^J |g_j|^2$, then for all $\omega \in \mathbb{T}^n$ where it is defined,

$$f(\omega) = \mathcal{Q}^{-1} \sum_{\gamma\in\Gamma^*} f^\gamma(\omega) = \mathcal{Q}^{-1} \sum_{j=1}^J \sum_{\gamma\in\Gamma^*} |g_j^\gamma(\omega)|^2 = \mathcal{Q}^{-1} \sum_{j=1}^J \sum_{\nu\in\Gamma} |(g_j)_\nu(\mathcal{M}^T\omega)|^2,$$

the last equality following from part (a) which was just proved. The last expression is a sum of $\mathcal{G}$-invariant squares, in light of Lemma 1.1, which completes the proof. $\square$

Using Lemma 1.2(a), the function $f(\tau;\omega)$ in (1.5) can be written in terms of the polyphase components of τ as well:

$$f(\tau;\omega) = 1 - \sum_{\nu\in\Gamma} |\tau_\nu(\mathcal{M}^T\omega)|^2. \tag{1.18}$$

1.6.2 Formally Real Fields

For some of our results, we will make use of the Artin-Schreier theory of formally real fields, as in [43]. We review the basic ideas we will require from this theory here, mostly without proof. A field K is *formally real* if it admits a total order. This is a relation $<$ on K satisfying the following properties:

(i) For $x \in K$, $0 < x$, $x = 0$, or $x < 0$, and not more than one of these holds.

(ii) If $x, y \in K$, and $0 < x$, $0 < y$, then $0 < x + y$ and $0 < xy$.

We will also sometimes use $x > 0$ to denote $0 < x$, and the symbols $\leq, \geq$ with their usual meanings.

Equivalently, a field K is formally real if -1 is not a sum of squares in K. Since we have $x^2 = (-x)^2$, and either $x > 0$ or $-x > 0$, by (ii), any nonzero square is necessarily positive in any ordering on K. Again by (ii), this holds for sums of squares also. On the other hand, an element of a formally real field which is positive in any ordering of K is called *totally positive*. This is an equivalence: $x \in K$ is totally positive if and only if x is a sum of squares in K [43, Ch. XI Cor. 2.3].

A field R is *real closed* if it is formally real, and any algebraic extension of R which is formally real must be equal to R. Every formally real field K is contained in a unique *real closure*, which is a real closed field which is an algebraic extension of K. If R is real closed, then R has a unique ordering with the positive elements given by the sums of squares in R. In fact, if $a \in R$, and $a > 0$, then $a = b^2$ for some $b \in R$. The idea of the proof is that $a > 0$ implies $R(\sqrt{a})$, which is an algebraic extension of R, is formally real, which means that $R(\sqrt{a}) = R$ since R is real closed, so a is a square of some element of R.

If R is real closed, then $R(\sqrt{-1})$ is algebraically closed.

Example 1.2 (Orderings and Real Closure)**.** Consider the field $\mathbb{Q}(\sqrt{2})$, which is an algebraic extension of $\mathbb{Q}$. We see that the only ordering possible on $\mathbb{Q}$ is the usual one, since $1 = 1^2 > 0$, and applying property (ii) of $<$ repeatedly, the positive elements in $\mathbb{Q}$ are exactly those having a representation as p/q for $p, q \in \mathbb{Z}$, $p, q > 0$. Then any ordering of $\mathbb{Q}(\sqrt{2})$ must extend this ordering (otherwise restricting the ordering on $\mathbb{Q}(\sqrt{2})$ to $\mathbb{Q}$ would result in a contradiction), and we see that there are two possible orderings for $\mathbb{Q}(\sqrt{2})$, where $\sqrt{2}$ takes the place of one of $\pm 1.414\ldots$ in the ordering. Indeed, if $\sqrt{2} > 0$, then given any $a \in \mathbb{Q}$ with $a > 0$ and $a^2 < 2$, $(\sqrt{2} - a)(\sqrt{2} + a) = 2 - a^2 > 0$, and since $\sqrt{2} + a > 0$ by (ii), $\sqrt{2} > a$. Arguing similarly for $b \in \mathbb{Q}$ with $b > 0$, $b^2 > 2$, we see that $\sqrt{2} = 1.414\ldots$. In the other ordering, $\sqrt{2} < 0$, and we can prove that $\sqrt{2} = -1.414\ldots$

Let R be the extension field of $\mathbb{Q}$ which is the maximal real subfield of $\mathbb{Q}^a$, the algebraic completion of $\mathbb{Q}$. Then R is the real closure of $\mathbb{Q}$, and we can choose an ordering on R which extends the ordering on $\mathbb{Q}(\sqrt{2})$. Supposing $\sqrt{2} > 0$, $\sqrt{2} = (\sqrt[4]{2})^2$, where $\sqrt[4]{2} \in R$. Moreover, in this setting, $R(\sqrt{-1}) = \mathbb{Q}^a$, so any element of $\mathbb{Q}^a$ may be written as $f + ig$, where $f, g \in R$, and $i = \sqrt{-1}$. $\square$

The real field we will primarily be interested in is $\mathbb{R}(\mathbf{x})$. In this case, the algebraic completion is the field K of Puiseux series, which are formal series of the form $x^{j/m} \sum_{k \geq_{\mathrm{e}} 0} c_k x^{k/m}$, where m is a positive integer, $j \in \mathbb{Z}^n$, and $c_k \in \mathbb{C}$ for all $k \in \mathbb{Z}^n$, $k \geq_{\mathrm{e}} 0$ [23, Cor. 13.15]. Since the real closure R of $\mathbb{R}(\mathbf{x})$ must satisfy $R(\sqrt{-1}) = K$, we see that R is just the field of such series with $c_k \in \mathbb{R}$ for all $k \in \mathbb{Z}^n$, $k \geq_{\mathrm{e}} 0$. While we will not need to use this explicit form for R, it may be helpful to have this in mind.

1.6.3 Results

In this section, we give several lemmas, which show that sos and sors representations exist in many settings. We start with the case of trigonometric polynomials, though we state these for ordinary polynomials being evaluated on $\partial\mathbb{D}$, which is equivalent (see Remark 1.1).

Result 1.8 (Sos Lemmata)**.** *(a) (Fejér-Riesz [14]) Let $f \in \mathbb{C}[z_1^{\pm 1}]$ be such that $f(z) \geq 0$ for all $z \in \partial\mathbb{D}$. Then $f(z) = |p(z)|^2$ for all $z \in \partial\mathbb{D}$, where $p \in \mathbb{C}[z_1]$.*

(b) (Scheiderer [50]) Let $f \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$ be such that $f(z) \geq 0$ for all $z \in (\partial\mathbb{D})^2$. Then f

has an sos representation on $(\partial\mathbb{D})^2$.

(c) (Charina et al.[11]) Let $n \geq 3$. *There is a nonnegative trigonometric polynomial in* n *variables with no sos representation.*

(d) (Dritschel [21]) Let $n \geq 2$, *and* $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ *be such that* $f(z) > 0$ *for all* $z \in (\partial\mathbb{D})^n$. *Then* f *has an sos representation on* $(\partial\mathbb{D})^n$.

Proof. (a) is proved for a special case in [14]. (b) may be found as Corollary 3.4 in [50] (see also [11, Theorem 2.4]). (c) is shown in [11]. (d) comes from [21], but the statement here comes from [25]. □

Moving away from the trigonometric polynomial case to that of ordinary polynomials with real coefficients, we have the following results, the first of which comes from Artin in 1927 [3] (see also [6]), but translated and adapted to our notation. The second comes from Pfister in 1967 [47] (see also [6]), and gives a bound on the number of squares in the representation guaranteed by Artin's Theorem. We will use these theorems and the following corollary in Chapter 4.

Result 1.9 (Sos Lemmata). *(a) (Artin [3]) Let* $f \in \mathbb{R}[\mathbf{x}]$ *be such that* $f(x) \geq 0$ *for all* $x \in \mathbb{R}^n$. *Then* f *is an sors on* $\mathbb{R}^n$ *of functions in* $\mathbb{R}(\mathbf{x})$, *i.e., there exists* $g_j \in \mathbb{R}(\mathbf{x}), 1 \leq j \leq J$ *such that for all* $x \in \mathbb{R}^n$, $f(x) = \sum_{j=1}^{J} g_j(x)^2$.

(b) (Pfister [47]) Let $f \in \mathbb{R}[\mathbf{x}]$ *be such that* $f(x) \geq 0$ *for all* $x \in \mathbb{R}^n$. *Then* f *is an sors on* $\mathbb{R}^n$ *of at most* 2^n *functions in* $\mathbb{R}(\mathbf{x})$, *i.e., there exist* $\{g_j\}_{j=1}^{J}$ *satisfying the conclusion of part (a) with* $J \leq 2^n$.

The following corollary combines parts (a) and (b) of the result above and extends to the case where f is a rational polynomial. The respective parts of this result were known to Artin and Pfister, but we show the proof because we will use this technique later.

Corollary 1.1 (Rational Case). *Let* $f \in \mathbb{R}(\mathbf{x})$, $f(x) \geq 0$ *for all* $x \in \mathbb{R}^n$ *at which it is defined. Then* f *is an sors on* $\mathbb{R}^n$ *of at most* 2^n *functions in* $\mathbb{R}(\mathbf{x})$.

Proof. Let $f = p/q$ be in lowest terms. Then $f = pq/q^2$, and since $q^2(x) \geq 0$ for all $x \in \mathbb{R}^n$, $(pq)(x) = f(x)q^2(x) \geq 0$ for all $x \in \mathbb{R}^n$. Then by Result 1.9(a) and (b), $pq = \sum_{j=1}^{J} g_j^2$, where $g_j \in \mathbb{R}(\mathbf{x})$ for all $1 \leq j \leq J \leq 2^n$, and thus $f = \sum_{j=1}^{J}(g_j/q)^2$, which is an sors representation with at most 2^n squares. $\square$

The following result is a natural extension of Artin's theorem, and was originally proved in [26]. If A is a matrix with entries in $\mathbb{R}(\mathbf{x})$, we say that it is *symmetric* when $A = A^T$, the ordinary transpose.

Result 1.10 (Gondard and Ribenboim)**.** *Let $A \in M_m(\mathbb{R}[\mathbf{x}])$ be a symmetric matrix. If $A(x)$ is positive semidefinite for all substitutions $x \in \mathbb{R}^n$, then A can be expressed as a sum of squares of symmetric matrices $B_j \in M_m(\mathbb{R}(\mathbf{x})), 1 \leq j \leq J$.*

In [48], Procesi and Schacher observed that the generators in this representation can be made to commute (also see [33]). In Section 4.2, we will show that the conclusion of this theorem is also true when $A \in \mathbb{C}(\mathbf{x})$, and we will further show that the result is true when the positive semidefiniteness assumptions hold for $x \in (\partial\mathbb{D})^n$.

1.7 Connection between Sums of Squares Representations and Extension Principles

In this section, we describe the existing results connecting sums of squares representations with the extension principles in Section 1.5. These connections are quite strong in both settings, but so far, this has only been thoroughly explored in the case of the UEP. We will give analogous results in the OEP setting in Chapter 5.

1.7.1 Unitary Extension Principle

In [11, 42], the authors make use of the UEP conditions in Result 1.6 to describe methods by which a multidimensional lowpass filter satisfying certain conditions may be used to create a multidimensional tight wavelet frame. These make use of the idea of finding an sos representation for the trigonometric polynomial in the sub-QMF condition (1.5). The following result is Theorem 2.2 in [11].

Result 1.11 (UEP and Sos Reps [11]). *Suppose τ is a lowpass mask that satisfies the sub-QMF condition, and $1-\sum_{\gamma\in\Gamma^*}|\tau(\omega+\gamma)|^2=\sum_{j=1}^J|g_j(\mathcal{M}^T\omega)|^2$, for all $\omega\in\mathbb{T}^n$. Then the $\mathcal{Q}+J$ functions*

$$q_{1,j}(\omega)=\tau(\omega)\overline{g_j(\mathcal{M}^T\omega)},\quad j=1,\ldots,J,$$

$$q_{2,\nu}(\omega)=\mathcal{Q}^{-1/2}e^{i\nu\cdot\omega}-\tau(\omega)\overline{\tau_\nu(\mathcal{M}^T\omega)},\quad \nu\in\Gamma,$$

satisfy the UEP conditions with τ, and thus form a tight wavelet filter bank with τ. Conversely, when τ satisfies the UEP conditions with some collection of highpass masks $\{q_\ell\}_{\ell=1}^r$, then $f(\tau;\cdot)$ has a $\mathcal{G}$-invariant sum of squares representation, and in particular, τ satisfies the sub-QMF condition.

We will use this result for the constructions considered in Chapters 2 and 3.

1.7.2 Vanishing Moments for UEP Construction

In this subsection, we analyze the vanishing moments of the highpass masks constructed in Result 1.11. While most of the results in this chapter are merely review, this section contains only new results.

Proposition 1.1. *In Result 1.11, let τ have accuracy number $a>0$ and flatness number b. Then the highpass masks $q_{1,j},1\le j\le J$ have exactly as many vanishing moments as $g_j,1\le j\le J$, and the highpass masks $q_{2,\nu}$ have at least* $\min\{a,b\}$ *vanishing moments.*

Proof. We consider the $q_{2,\nu},\nu\in\Gamma$ first. Let $m=\min\{a,b\}$. The result is clear when $m=1$, so let $m\ge 2$. If β is a multiindex with $1\le|\beta|\le m-1$, then by the assumptions on the accuracy and flatness numbers of τ,

$$D^\beta\tau(\gamma)=0 \text{ for all } \gamma\in\Gamma^*. \tag{1.19}$$

Now we compute $D^\alpha q_{2,\nu}(\omega)$ for some α with $1\le|\alpha|\le m-1$. Recalling that we are assuming all lowpass filters have real coefficients, by (1.12), we obtain

$$\frac{i^{|\alpha|}\nu^\alpha}{\mathcal{Q}^{1/2}}\exp(i\nu\cdot\omega)-\sum_{0\le_e\beta\le_e}\binom{\alpha}{\beta}D^\beta\tau(\omega)\overline{D^{\alpha-\beta}[\tau_\nu(\mathcal{M}^T\omega)]},$$

and at $\omega = 0$, this yields $\mathcal{Q}^{-1/2} i^{|\alpha|} \nu^\alpha - \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \tau(0) \overline{D^{\alpha-\beta}[\tau_\nu(\mathcal{M}^T\omega)]_{\omega=0}}$. By (1.19), only the term with $\beta = 0$ remains in this sum. Since $\tau(0) = 1$, this yields

$$D^\alpha q_{2,\nu}(0) = \mathcal{Q}^{-1/2} i^{|\alpha|} \nu^\alpha - \overline{D^\alpha[\tau_\nu(\mathcal{M}^T\omega)]_{\omega=0}}. \tag{1.20}$$

By (1.7), and using (1.12) again, we see that $D^\alpha[\tau_\nu(\mathcal{M}^T\omega)]$ equals

$$\mathcal{Q}^{-1/2} \sum_{\gamma \in \Gamma^*} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \tau(\omega + \gamma)(-i)^{|\alpha-\beta|} \nu^{\alpha-\beta} \exp(-i\nu \cdot (\omega + \gamma)),$$

and at $\omega = 0$, by (1.19) and the positive accuracy of τ, the only nonzero term in the sum is when $\beta = 0$ and $\gamma = 0$, which yields $D^\alpha[\tau_\nu(\mathcal{M}^T\omega)]_{\omega=0} = \mathcal{Q}^{-1/2}(-i)^{|\alpha|}\nu^\alpha$. From (1.20), we see that $D^\alpha(q_{2,\nu})(0) = 0$ whenever $|\alpha| \le m - 1$, and thus $q_{2,\nu}$ has at least m vanishing moments, as desired.

The analysis of the highpass masks $q_{1,j}$ is simpler: since for $\omega \approx 0$, $\tau(\omega) \approx 1$, we see that $g_j(\omega) = O(\|\omega\|^l)$ if and only if $q_{1,j}(\omega) = O(\|\omega\|^l)$. □

We may obtain more detailed information about the vanishing moments of the sos generators g_j by taking a closer look at the relation $f(\tau;\cdot) = \sum_{j=1}^J |g_j(\mathcal{M}^T \cdot)|^2$. Notice from the definition of $f(\tau;\cdot)$ in (1.5), we have $f(\tau;0) = 0$ for any lowpass mask τ with positive accuracy, hence $f(\tau;\cdot)$ can be considered as a highpass mask in this case. In fact, we have the following result.

Proposition 1.2. *Let τ be a lowpass mask with accuracy number $a > 0$ and flatness number b. Let $1 - |\tau|^2$ have $2c$ vanishing moments. Then $f(\tau;\cdot)$ has at least $\min\{2a, 2c\} \ge \min\{2a, b\}$ vanishing moments.*

Proof. By the assumptions on τ, we see that for $\omega \approx 0$, $\tau(\omega) = 1 + O(\|\omega\|^b)$, and for all $\gamma \in \Gamma^* \setminus \{0\}$, $\tau(\omega + \gamma) = O(\|\omega\|^a)$. Expanding the square, $|\tau(\omega)|^2 = 1 + O(\|\omega\|^b)$, which shows that $2c \ge b$, and clearly $|\tau(\omega+\gamma)|^2 = O(\|\omega\|^{2a})$. This gives for $\omega \approx 0$,

$$f(\tau;\omega) = 1 - \sum_{\gamma \in \Gamma^*} |\tau(\omega + \gamma)|^2 = O(\|\omega\|^{2c}) + O(\|\omega\|^{2a}),$$

which completes the proof. □

These results lead to the following theorem, which gives lower bounds on the vanishing moments of the highpass masks constructed from Result 1.11 in terms of the flatness and accuracy numbers of the lowpass mask τ. Note that as a special case, when τ is an interpolatory lowpass mask, the flatness number is always at least as large as the accuracy number.

Theorem 1.1 (VMs for Highpass Masks in Result 1.11)**.** *Let τ be a lowpass mask with accuracy number $a > 0$ and flatness number b, and let $1 - |\tau|^2$ have $2c$ vanishing moments. If $f(\tau; \cdot)$ has an sos representation with trigonometric polynomials $g_j, 1 \leq j \leq N$, then each g_j has at least $\min\{a, c\} \geq \lceil \min\{2a, b\}/2 \rceil$ vanishing moments. Therefore, for the highpass masks in Result 1.11, $q_{1,j}, 1 \leq j \leq J$ have at least $\lceil \min\{2a, b\}/2 \rceil$ vanishing moments, and $q_{2,\nu}, \nu \in \Gamma$ have at least $\min\{a, b\}$ vanishing moments.*

Proof. By Proposition 1.2, $\sum_{j=1}^{J} |g_j(\mathcal{M}^T\omega)|^2 = f(\tau; \omega) = O(\|\omega\|^{\min\{2a,2c\}})$ for $\omega \approx 0$. Thus all of the summands $|g_j(\omega)|^2 = O(\|\omega\|^{\min\{2a,2c\}})$ for $\omega \approx 0$, so $g_j(\omega) = O(\|\omega\|^{\min\{a,c\}})$ for all $1 \leq j \leq J$, and these all have at least $\lceil \min\{2a, b\}/2 \rceil$ vanishing moments, using the proof of Proposition 1.2 to see that $2c \geq b$. The remaining statements follow from Proposition 1.1. $\square$

We construct highpass filters with exactly this many vanishing moments in Examples 2.2, 3.3, and 3.4.

1.7.3 Oblique Extension Principle

In [42], a construction based on the OEP and sors representations was given which may be used to construct highpass masks with maximum vanishing moments. This was done in the case of dyadic dilation, and supposes that S is a rational trigonometric polynomial satisfying

$$\frac{1}{S(2\omega)} - \sum_{\gamma \in \{0,\pi\}^n} \frac{|\tau(\omega+\gamma)|^2}{S(\omega+\gamma)} \equiv 0, \tag{1.21}$$

as well as the necessary condition $S(0) = 1$, and an sors condition

$$1/S(\omega) = \sum_{k=1}^{K} |s_k(\omega)|^2, \tag{1.22}$$

for some rational trigonometric polynomials s_k. In this setting the following result holds, which is Theorem 6.1 in [42]:

Result 1.12 (OEP and Sos Reps [42])**.** *Let τ be a trigonometric polynomial lowpass mask. Suppose S is a rational trigonometric polynomial with $S(0) = 1$, satisfying (1.21) and (1.22). Then there exist $2^n K$ rational trigonometric polynomial wavelet masks with maximum vanishing moments satisfying the OEP conditions (1.16) with S and τ for all $\omega \in \mathbb{T}^n, \gamma \in \{0, \pi\}^n$.*

We view Equation (1.21) as an "oblique QMF condition," generalizing the QMF condition to allow a vmr function S. In Chapter 5, we will see that considering the corresponding "oblique sub-QMF condition," we obtain an equivalence with the existence of highpass masks satisfying the OEP conditions with this S and τ, analogous to Result 1.11. In addition, we find assumptions on S which ensure that all of the conditions of Result 1.7 hold, which means that the highpass masks will generate a tight wavelet frame for $L^2(\mathbb{R}^n)$.

For now, however, we turn to our UEP-based constructions using the coset sum method.

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Chapter 2

Coset Sum Lowpass Masks

In this chapter, we discuss constructions of tight wavelet frames using lowpass masks obtained from the coset sum method [38]. The goal of the coset sum method is to construct nonseparable lowpass masks in any dimension $n \geq 2$ using univariate lowpass masks as input, with properties of the input mask being carried over to the constructed multidimensional mask, particularly the accuracy number and interpolatory property. As a result, the constructions in this chapter enable one to choose a univariate lowpass mask and construct a nonseparable multidimensional tight wavelet frame, when the appropriate conditions on the input mask are satisfied.

The constructions described here may be broken into the cases of interpolatory input masks and non-interpolatory input masks. In the first case, we are able to find the sums of squares representations necessary for our construction by simply applying the Fejér-Riesz Lemma to a collection of univariate trigonometric polynomials. In the non-interpolatory case, the trigonometric polynomial which is required to have a sum of squares representation does not split nicely into a collection of nonnegative univariate trigonometric polynomials, so we are not able to apply the Fejér-Riesz Lemma, but we construct a positive semidefinite matrix P and vector of complex exponential functions x so that the trigonometric polynomial we wish to write as a sum of squares may be written as x^*Px. Then, factorizing P as AA^* for some matrix A, we obtain the desired sum of squares representation, since $x^*AA^*x = \sum_{j=1}^{J} |A_j^*x|^2$, where A_j is the jth column of the matrix A. In order to prove that this construction succeeds, we do require some conditions on the input lowpass mask, but we are still able to find a reasonably large collection of examples satisfying these conditions.

Throughout this chapter, we consider the setting of dyadic dilation, so that $\mathcal{M} = 2I$, where I is the $n \times n$ identity matrix. For definiteness, we will assume that $\Gamma^* = \{0, \pi\}^n$ and $\Gamma = \{0, 1\}^n$, though the specific choice of cosets is immaterial for our results.

In Section 2.1, we describe our methods for constructing sos representations for $f(\tau; \omega)$ arising from coset sum generated lowpass masks τ, which in turn provide construction methods for tight wavelet frames with the lowpass mask τ. In Section 2.2, we give several examples of our construction methods, and concluding remarks are given in Section 2.3.

Most of the results appearing in this chapter were first published in [35].

2.1 Coset Sum Tight Wavelet Frames: Theory and Constructions

In this section, we begin by computing the function $f(\tau; \cdot)$ when τ is a lowpass mask obtained from the coset sum operator. In Section 2.1.2, we find an sos representation for $f(\tau; \cdot)$ in the case that τ is interpolatory, which relies only on the Fejér-Riesz Lemma. Next, we give a first matrix-based method for finding an sos representation for a nonnegative trigonometric polynomial g, which proceeds in three steps: first, we find a Hermitian matrix P_1 and vector of complex exponential functions x such that $g = x^* P_1 x$. Next, we attempt to modify the diagonal entries of P_1 to change P_1 into a positive semidefinite matrix P that still satisfies the equality $g = x^* P x$. An sos representation may then be found by factorizing P as AA^* for some matrix A, giving sos generators $A^* x$.

In Section 2.1.4, we apply the ideas of the previous section to the case where $g = f(\tau; \cdot)$, for τ a lowpass mask coming from the coset sum operator. Using the structure of lowpass masks coming from this operator, we are able to significantly reduce the size of the matrix compared to the one we would get from the first approach. This mostly comes from designing P_1 in a much more space-efficient way than the first method, and then modifying the diagonal entries as before. We are also able to give conditions on the input lowpass mask under which the constructed matrix is guaranteed to be positive semidefinite, and we discuss how these compare to the assumptions that $f(\tau; \cdot)$ has an sos representation, or is only nonnegative. We also discuss some simple sufficient conditions for the construction to succeed.

2.1.1 $f(\tau;\cdot)$ for Coset Sum Generated Lowpass Masks τ

To compute $f(\tau;\cdot)$, we would typically start from the definition given in Equation (1.5). However, the structure imparted on τ by the coset sum operator appears less obviously in the trigonometric polynomials τ^γ, $\gamma \in \Gamma^*$ than it does in the polyphase components τ_ν, $\nu \in \Gamma$. As such, we will proceed using the formula given in Equation (1.18):

$$f(\tau;\omega) = 1 - \sum_{\nu\in\Gamma} |\tau_\nu(2\omega)|^2.$$

Now, we find the polyphase components of τ coming from the coset sum operator.

From the formula in Equation (1.9), we see that we may write τ_ν as a trigonometric polynomial whose coefficients are supported on the set $\{2k-\nu : k \in \mathbb{Z}^n\} \cap \operatorname{supp}(h)$, where h is the filter associated with τ. From the definition of the coset sum operator (see the discussion in Section 1.4.1), $\operatorname{supp}(h) = \bigcup_{\nu\in\Gamma'} \operatorname{span}(\nu)$. Since $2k - \nu_1 = \ell\nu_2$ for some $k \in \mathbb{Z}^n$, $\ell \in \mathbb{Z}$, $\nu_1, \nu_2 \in \Gamma$, if and only if $\nu_1 \equiv \ell\nu_2 \pmod{2\mathbb{Z}^n}$, we see that the only coefficients $h(\ell\nu_2)$ which are nonzero in the polyphase component τ_{ν_1} appear when either $\nu_1 = 0$ and $\ell \in 2\mathbb{Z}$, or $\nu_1 \neq 0$ and $\nu_2 = \nu_1$, $\ell \in (2\mathbb{Z}+1)$.

Then for $\nu = 0$, since $h(2k\nu') = H(2k)$ for $\nu' \in \Gamma'$, whenever $k \neq 0$, we have

$$\tau_0(\omega) = \frac{1}{2^{n/2}} \left(2 - 2^n + (2^n - 1)H(0) + \sum_{\nu\in\Gamma'} \sum_{k\in\mathbb{Z}\setminus\{0\}} H(2k) e^{-ik\nu\cdot\omega} \right).$$

For $\nu \in \Gamma'$, using the computation above and letting $\ell = 2j-1$:

$$\sum_{k\in\mathbb{Z}^n} h(2k-\nu) e^{-ik\cdot\omega} = \sum_{j\in\mathbb{Z}} h((2j-1)\nu) e^{-ij\nu\cdot\omega} = \sum_{j\in\mathbb{Z}} H(2j-1) e^{-ij\nu\cdot\omega},$$

hence we have, for $\nu \in \Gamma'$,

$$\tau_\nu(\omega) = \frac{1}{2^{n/2}} \sum_{j\in\mathbb{Z}} H(2j-1) e^{-ij\nu\cdot\omega}. \tag{2.1}$$

Now we are ready to write $f(\tau;\cdot)$ for the coset sum generated lowpass filter τ. We have

$$
\begin{aligned}
&f(\tau;\omega)\\
&= 1-\frac{(2-2^n+(2^n-1)H(0))^2}{2^n}-\frac{2-2^n+(2^n-1)H(0)}{2^{n-1}}\sum_{\nu\in\Gamma'}\sum_{k\in\mathbb{Z}\setminus\{0\}}H(2k)\cos(2k\nu\cdot\omega)\\
&-\frac{1}{2^n}\sum_{\nu,\eta\in\Gamma'}\sum_{j,k\in\mathbb{Z}\setminus\{0\}}H(2k)H(2j)e^{-2i(k\nu-j\eta)\cdot\omega}-\frac{1}{2^n}\sum_{\nu\in\Gamma'}\sum_{j,k\in\mathbb{Z}}H(2k-1)H(2j-1)e^{-2i(k-j)\nu\cdot\omega}\\
&= 1-\frac{(2-2^n+(2^n-1)H(0))^2}{2^n}-\frac{2^n-1}{2^n}\sum_{j\in\mathbb{Z}\setminus\{0\}}H(2j)^2-\frac{2^n-1}{2^n}\sum_{j\in\mathbb{Z}}H(2j-1)^2\\
&-\frac{1}{2^{n-1}}\sum_{\nu\in\Gamma'}\sum_{k\geq 1}(2-2^n+(2^n-1)H(0))(H(-2k)+H(2k))\cos(2k\nu\cdot\omega)\\
&-\frac{1}{2^{n-1}}\sum_{\nu\in\Gamma'}\sum_{k\geq 1}\left[\sum_{j\neq 0,-k}H(2j)H(2(j+k))+\sum_{j\in\mathbb{Z}}H(2j-1)H(2(j+k)-1)\right]\cos(2k\nu\cdot\omega)\\
&-\frac{1}{2^{n-1}}\sum_{\substack{\nu,\eta\in\Gamma'\\ \nu\geq_{\mathrm{lex}}\eta}}\sum_{j,k\in\mathbb{Z}\setminus\{0\}}H(2k)H(2j)\cos(2(k\nu-j\eta)\cdot\omega).
\end{aligned}
$$

Then we may write $f(\tau;\omega)$ as

$$
\alpha-\sum_{\nu\in\Gamma'}\sum_{k\geq 1}\alpha(k)\cos(2k\nu\cdot\omega)-\frac{2}{2^n}\sum_{\substack{\nu,\eta\in\Gamma'\\ \nu\geq_{\mathrm{lex}}\eta}}\sum_{j,k\in\mathbb{Z}\setminus\{0\}}H(2k)H(2j)\cos(2(k\nu-j\eta)\cdot\omega),
$$

where

$$
\alpha = 1-\frac{(2-2^n+(2^n-1)H(0))^2}{2^n}-\frac{2^n-1}{2^n}\sum_{j\in\mathbb{Z}\setminus\{0\}}H(j)^2, \tag{2.2}
$$

and for $k\geq 1$,

$$
2^{n-1}\alpha(k) = (2^n-2)(H(0)-1)(H(2k)+H(-2k))+\sum_{j\in\mathbb{Z}}H(j)H(j+2k), \tag{2.3}
$$

where we added and subtracted $H(0)(H(2k)+H(-2k))$ to get the form of $\alpha(k)$ written here.

2.1.2 Sos Representations from the Fejér-Riesz Lemma: Interpolatory Input Masks

From Result 1.4, the coset sum operator $\mathcal{C}_n$ preserves the positive accuracy and the interpolatory properties. We will soon show that when the input univariate lowpass mask R is interpolatory, so that τ is also, $f(\tau;\cdot)$ splits into a sum of univariate trigonometric polynomials which are nonnegative precisely when R satisfies the univariate sub-QMF condition. Then applying the Fejér-Riesz Lemma (c.f. Result 1.8(a)), we obtain an sos representation for $f(\tau;\cdot)$, so we may obtain the highpass masks generating a tight wavelet frame for this lowpass mask using the construction of Result 1.11.

Theorem 2.1. *Let R be a univariate, positive accuracy, interpolatory lowpass mask with corresponding filter H, such that R satisfies the sub-QMF condition, and $R(0) = 1$. Let τ be the output of the coset sum with input R into n dimensions, for some $n \geq 2$. Then $f(\tau;\omega) = 1 - \sum_{\nu\in\Gamma} |\tau_\nu(2\omega)|^2$ has an sos representation, and there is a tight wavelet filter bank with τ having $2^{n+1} - 1$ highpass filters.*

Proof. By the interpolatory condition, $R_0(\omega) = 2^{-1/2}$, and the sub-QMF condition for R gives

$$f(R;\omega) = \frac{1}{2}\left(1 - \left|\sum_k H(2k-1)e^{-2ik\omega}\right|^2\right) \geq 0, \quad \text{for } \omega \in \mathbb{T}.$$

Then if τ is the output of the coset sum into n dimensions, τ must also be interpolatory and have positive accuracy, so $\tau_0(\omega) = 2^{-n/2}$. Now we have from Equation (2.1):

$$\begin{aligned} f(\tau;\omega) &= 1 - 2^{-n} - 2^{-n}\sum_{\nu\in\Gamma'}\left|\sum_k H(2k-1)e^{-2ik\nu\cdot\omega}\right|^2 \\ &= \frac{1}{2^n}\sum_{\nu\in\Gamma'}\left(1 - \left|\sum_k H(2k-1)e^{-2ik\nu\cdot\omega}\right|^2\right) \\ &= \frac{2}{2^n}\sum_{\nu\in\Gamma'} f(R;\nu\cdot\omega), \quad \text{for } \omega \in \mathbb{T}^n. \end{aligned} \tag{2.4}$$

Since we have $f(R;\omega) \geq 0$ for $\omega \in \mathbb{T}$ from above, by the Fejér-Riesz Lemma, since $f(R;\omega/2)$ is a nonnegative trigonometric polynomial, $f(R;\omega/2) = |p(\omega)|^2$, $\omega \in \mathbb{T}$, for some trigonometric polynomial p, so we have that

$$f(\tau;\omega) = \sum_{\nu\in\Gamma'} \left| \sqrt{\frac{2}{2^n}} p(2\nu\cdot\omega) \right|^2, \quad \text{for } \omega \in \mathbb{T}^n,$$

which is an sos representation for $f(\tau;\cdot)$ with $2^n - 1$ sos generators. That there exists a tight wavelet filter bank with τ as the lowpass mask is then the content of Result 1.11. $\square$

2.1.3 Sos Representations from Matrix Factorizations: First Approach

In this subsection we first make some observations about sos representations from positive semidefinite matrices for general nonnegative trigonometric polynomials, and then apply them for the special case when the polynomial is $f(\tau;\cdot)$.

We observe that a nonnegative trigonometric polynomial g has an sos representation if and only if there exists a positive semidefinite matrix P and a vector $x = [e^{-ik\cdot\omega}]_{k\in\mathcal{I}}$, for some finite set $\mathcal{I} \subseteq \mathbb{Z}^n$, such that $g = x^*Px$. Indeed, given such a P, the Cholesky factorization of $P = LL^*$ for a lower triangular matrix L (or, indeed any representation of the form $P = AA^*$ for a matrix A), gives an sos representation of g with generators L^*x (each entry of which is seen to be a trigonometric polynomial), since $g = x^*Px = (L^*x)^*(L^*x)$. Conversely, given an sos representation of g as in Equation (1.17), if for each j, $g_j(\omega) = \sum_{k\in\mathbb{Z}^n} c_{j,k}e^{-ik\cdot\omega}$, and we let $\mathcal{I} = \bigcup_{j=1}^{J}\{k \in \mathbb{Z}^n : c_{j,k} \neq 0\} \cup \{0\}$ with some ordering, then we may form the matrix A of size $|\mathcal{I}| \times J$, with $A_{k,j} = \overline{c_{j,k}}$ for $1 \leq j \leq M$, $k \in \mathcal{I}$, and this gives the representation $g = x^*(AA^*)x$ for this A and $x = [e^{-ik\cdot\omega}]_{k\in\mathcal{I}}$, where clearly AA^* is positive semidefinite.

Remark 2.1. Let g be a nonnegative trigonometric polynomial, such that $g = \sum_k c_k e^{-ik\cdot\omega}$, with real coefficients c_k, $k \in \mathbb{Z}^n$. Let $x = [e^{-ik\cdot\omega}]$, for some ordering of the set $\{k \in \mathbb{Z}^n : k \geq_{\text{lex}} 0, c_k \neq 0\} \cup \{0\}$ with 0 as the last entry. Observe that because g is real-valued and has real coefficients, $c_k = c_{-k}$. Consider the matrix P_1, with nonzero entries only in its last row and column, and indexed in the same way as x, such that $(P_1)_{0,0} = c_0$, and $(P_1)_{0,k} = (P_1)_{k,0} = c_k$. Using lines to separate the different regions of these matrices, we have

$$P_1 = \left[\begin{array}{ccc|c} \ddots & \ddots & \vdots & \vdots \\ \ddots & 0 & \ddots & c_k \\ \cdots & \ddots & \ddots & \vdots \\ \hline \cdots & c_k & \cdots & c_0 \end{array}\right], \quad x = \left[\begin{array}{c} \vdots \\ e^{-ik\cdot\omega} \\ \vdots \\ \hline 1 \end{array}\right].$$

Clearly, $g = x^* P_1 x$, but typically, P_1 will not be positive semidefinite. To see this, suppose for example that c_0 and c_k are both nonzero for some $k >_{\text{lex}} 0$, and $c_0 > 0$. Let y be the vector which has 0 in every entry except at the entries indexed by k and 0, where it is equal to $-1/c_k$ and $1/c_0$, respectively. Then restricting to this submatrix of P_1, we have

$$y^* P_1 y = [-1/c_k \; 1/c_0] \begin{bmatrix} 0 & c_k \\ c_k & c_0 \end{bmatrix} \begin{bmatrix} -1/c_k \\ 1/c_0 \end{bmatrix} = 0 - 1/c_0 - 1/c_0 + 1/c_0 = -1/c_0 < 0.$$

To remedy this situation, consider P_2, which has the same last row and column as P_1 except $(P_2)_{0,0}$, but $(P_2)_{0,0} = c_0 - \sum_{k>_{\text{lex}}0} |c_k|$, $(P_2)_{k,k} = |c_k|$ for $k >_{\text{lex}} 0$, and $(P_2)_{j,k} = 0$ elsewhere. This gives

$$P_2 = \left[\begin{array}{cccc|c} |c_{k_1}| & 0 & \ddots & \vdots & c_{k_1} \\ 0 & \ddots & 0 & \ddots & \vdots \\ \ddots & 0 & |c_{k_2}| & \ddots & c_{k_2} \\ \cdots & \ddots & \ddots & \ddots & \vdots \\ \hline c_{k_1} & \cdots & c_{k_2} & \cdots & c_0 - \sum_{k>_{\text{lex}}0} |c_k| \end{array}\right].$$

Then it is clear that P_2 again satisfies $g = x^* P_2 x$ for the vector x above, and is weakly diagonally dominant with nonnegative diagonal entries in (at least) all but the last row. If it happens that $(P_2)_{0,0} \geq \sum_{k>_{\text{lex}}0} |c_k|$, then this holds for the last row as well, which implies that P_2 is positive semidefinite. Put differently, if it happens that $c_0 \geq \sum_{k\neq 0} |c_k|$, the matrix P constructed in this way will be weakly diagonally dominant and positive semidefinite. $\square$

The following simple lemma formalizes the idea from the remark above, namely redistributing the constant term of the trigonometric polynomial g along the diagonal in an effort to make the matrix P positive semidefinite, as in the change from P_1 to P_2. A more general version of this idea is found in [44].

Lemma 2.1. [Change of Diagonal] *Let g be a nonnegative trigonometric polynomial, such that $g(\omega) = \sum_{k\in\mathbb{Z}^n} c_k e^{-ik\cdot\omega}$, with real coefficients c_k. Let $J = \{k \in \mathbb{Z}^n : c_k \neq 0\} \cup \{0\}$ with some ordering, and for some nonempty $S \subseteq J$ with the inherited ordering, let $x = [e^{-ik\cdot\omega}]_{k\in S}$, and $P \in M_{|S|}(\mathbb{R})$ be a Hermitian matrix such that $g = x^*Px$. Suppose that a diagonal matrix $D \in M_{|S|}(\mathbb{R})$ satisfies $\sum_{i\in S} D_{i,i} = 0$ and $P_{i,i} + D_{i,i} \geq \sum_{j\in S, j\neq i} |P_{i,j}|$ for all $i \in S$. Then $P + D$ is positive semidefinite (and weakly diagonally dominant), and g has a sos representation.*

By choosing the trigonometric polynomial g in Remark 2.1 as $f(\tau;\cdot)$, with τ a lowpass mask output by the coset sum method that satisfies the sub-QMF condition $f(\tau;\cdot) \geq 0$, after applying Result 1.11, we obtain the *naive construction method* for tight wavelet filter banks with coset sum lowpass masks. In the next section, we introduce a more sophisticated method than this one for generating a matrix P and vector x satisfying $f(\tau;\cdot) = x^*Px$. Without assuming any special structure for the input mask to the coset sum, the method described in Section 2.1.4 will typically result in a significantly smaller matrix than the one described here, though the sos generators are likely to be more complicated. Depending on the preferences of the filter designer, then, it may be beneficial to compare these approaches to obtaining the sos representation of f and the resulting frames. The naive method described here will typically result in many more sos generators (and thus wavelet masks), which have a simple form if the Cholesky factorization is used. The method of Theorem 2.2 in Section 2.1.4 will typically result in far fewer sos generators, but these may be more complicated.

2.1.4 Sos Representations from Matrix Factorizations: General Input Masks

In the theorem below, we provide a condition (i.e. Condition ($\diamondsuit$)) on the univariate mask R for the existence of a positive semidefinite matrix P and a vector of complex exponentials x such that $f(\tau;\cdot) = x^*Px$ when $\tau = \mathcal{C}_n[R]$ is the coset sum lowpass mask generated from R, which in turn implies that $f(\tau;\cdot)$ has an sos representation. In Remark 2.4, we discuss the relationships between Condition ($\diamondsuit$), the sub-QMF condition for τ, the sub-QMF condition for R, and the existence of an sos representation for $f(\tau;\cdot)$.

Theorem 2.2. *Let R be a positive accuracy mask with lowpass filter H, such that $R(0) = 1$, and let n be an integer at least 2. Let τ be the output of the coset sum method into n dimensions with input R, and let $f(\tau;\omega) = 1 - \sum_{\nu\in\Gamma} |\tau_\nu(2\omega)|^2$. If H satisfies the following condition:*

$$\alpha(k) \geq 0 \text{ for all } k \geq 1, \text{ and } H(2k)H(2j) \geq 0 \text{ for all } k, j \in \mathbb{Z} \setminus \{0\}, \tag{$\diamondsuit$}$$

where $\alpha(k)$ is defined as in (2.3), then $f = x^*Px$ for a vector of complex exponentials x, where P is positive semidefinite (and weakly diagonally dominant), and thus a tight wavelet filter bank exists with τ as the lowpass mask.

Proof. We begin by constructing a Hermitian matrix Q and a vector of complex exponentials x such that $x^*Qx = f$. Let

$$N = \min\{2l : l \in \mathbb{Z}, l \geq 0, H(k) = H(-k) = 0 \text{ for all } k > 2l\}. \tag{2.5}$$

Let $J = \{(0,0)\} \cup \{(\nu,k) : \nu \in \Gamma', k \in \{-N/2,\dots,N/2\} \setminus \{0\}\}$, ordered in blocks $(\nu,-N/2)$, $\dots$, $(\nu,-1)$, $(\nu,1)$, $\dots$, $(\nu,N/2)$, for $\nu \in \Gamma'$ in some ordering, with $(0,0)$ as the last element. Let $x = [e^{-2ik\nu\cdot\omega}]_{(\nu,k)\in J}$, and for $(\eta,j),(\nu,k) \in J$, let

$$Q_{(\eta,j),(\nu,k)} = \begin{cases} \alpha & \nu = \eta = 0 \\ -\alpha(k)/2 & \nu \neq 0, \eta = 0, k > 0 \\ -\alpha(j)/2 & \eta \neq 0, \nu = 0, j > 0 \\ -2^{-n}H(2k)H(2j) & \nu,\eta \in \Gamma', \nu \neq \eta \\ -\alpha(N/2-j)/2 & \nu = \eta \in \Gamma', k = N/2, j < 0 \\ -\alpha(N/2-k)/2 & \nu = \eta \in \Gamma', j = N/2, k < 0 \\ 0 & \text{otherwise,} \end{cases}$$

where α and $\alpha(k)$ are defined as in (2.2) and (2.3), respectively. By inspection of the product, we see that $f = x^*Qx$. We now apply Lemma 2.1 to Q to obtain the matrix $P = Q + D$, where as in Remark 2.1, we add the sum of the magnitudes of the off-diagonal entries to each of the diagonal entries in all but the last row, and subtract the sum of these new diagonal entries from the last diagonal entry. That is, we let D be the diagonal matrix with, for $(\nu, k) \in J$, $D_{(\nu,k),(\nu,k)} = \beta(k)$ for $\nu \neq 0$, and $D_{(0,0),(0,0)} = -(2^n - 1)\sum_{s=-N/2,s\neq 0}^{N/2} \beta(s)$, where

$$\beta(k) = \begin{cases} \frac{2^n-2}{2^n}|H(2k)|\sum_{j=-N/2,j\neq 0}^{N/2}|H(2j)| + |\alpha(N/2-k)|/2 & \text{if } k < 0, \\ \frac{2^n-2}{2^n}|H(2k)|\sum_{j=-N/2,j\neq 0}^{N/2}|H(2j)| + |\alpha(k)|/2 & \text{if } 0 < k < N/2, \\ \frac{2^n-2}{2^n}|H(N)|\sum_{j=-N/2,j\neq 0}^{N/2}|H(2j)| + \sum_{j=N/2}^{N}|\alpha(j)|/2 & \text{if } k = N/2. \end{cases}$$

Let $P = Q + D$. Then by Lemma 2.1, P also satisfies $x^*Px = f$, and it remains to check that Condition ($\diamondsuit$) implies the positive semidefiniteness and weak diagonal dominance of P.

We see that for $\nu \neq 0$, $P_{(\nu,k),(\nu,k)} = D_{(\nu,k),(\nu,k)} = \sum_{(\eta,j)\neq(\nu,k)} |Q_{(\nu,k),(\eta,j)}|$. The equality $P_{(\nu,k),(\nu,k)} = \sum_{(\eta,j)\neq(\nu,k)} |Q_{(\nu,k),(\eta,j)}|$ holds when $\nu = 0$ as well if we have

$$\alpha - (2^n - 1)\sum_{\substack{k=-N/2 \\ k\neq 0}}^{N/2} \beta(k) = \frac{2^n-1}{2}\sum_{k=1}^{N/2}|\alpha(k)|.$$

By Condition ($\diamondsuit$), this is equivalent to:

$$\alpha - \frac{(2^n-1)(2^n-2)}{2^n}\sum_{\substack{j,k=-N/2 \\ j,k\neq 0}}^{N/2} H(2k)H(2j) - (2^n-1)\sum_{k=1}^{N}\alpha(k) = 0,$$

the left hand side of which is just $f(0)$, which equals 0 by the positive accuracy condition. Thus, we can apply the last part of Lemma 2.1 to say that P is positive semidefinite, and f has an sos representation. That a tight wavelet filter bank exists with τ as the lowpass mask is then the content of Result 1.11. $\square$

Remark 2.2. The matrix P in the proof clearly has a block matrix structure. More precisely, we define a vector $v \in \mathbb{R}^N$ of length N as in Equation (2.5), and Hermitian matrices $B, C \in M_N(\mathbb{R})$ of order N as

$$v = [0, \cdots, 0, -\alpha(1)/2, \cdots, -\alpha(N/2)/2]$$

$$B = \begin{bmatrix} \beta(-N/2) & & & & & -\alpha(N)/2 \\ & \ddots & & & & \vdots \\ & & \beta(-1) & & & -\alpha(N/2+1)/2 \\ & & & \beta(1) & & \\ & & & & \ddots & \\ -\alpha(N)/2 & \cdots & -\alpha(N/2+1)/2 & & & \beta(N/2) \end{bmatrix}$$

$$C = -\frac{1}{2^n} \begin{bmatrix} H(-N)^2 & H(-N)H(2-N) & \cdots & H(-N)H(N) \\ H(2-N)H(-N) & H(2-N)^2 & \cdots & H(2-N)H(N) \\ \vdots & \vdots & \ddots & \vdots \\ H(N)H(-N) & H(N)H(2-N) & \cdots & H(N)^2 \end{bmatrix},$$

where H is the univariate lowpass filter, and $\alpha(k)$ and $\beta(k)$ are the parameters determined by H as in the proof. Note that for the matrix C the zero index is absent, so in the first row (or column) we have $H(-N)H(-2)$ followed by $H(-N)H(2)$. Then the matrix P is given as

$$P = \begin{bmatrix} B & C & C & \cdots & C & v^T \\ C & B & C & \cdots & C & v^T \\ C & C & B & \ddots & C & v^T \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ C & C & C & \cdots & B & v^T \\ v & v & v & \cdots & v & b \end{bmatrix}, \tag{2.6}$$

where under Condition ($\Diamond$), $b = \frac{2^n-1}{2} \sum_{k=1}^{N/2} |\alpha(k)|$. $\square$

Remark 2.3. In the above proof and Remark 2.2, we chose N to be even, since this makes the indexing of the matrix P simpler, but for some filters, this choice essentially corresponds to zero-padding the outside of the filter to extend the support. The effect of this on the matrix P is that zero rows and columns may appear for certain filter inputs, and these indices can be removed from the index set J (and the corresponding rows and columns from P and x) with no change to the equality $f = x^*Px$, since the zero rows and columns of P do not contribute to this product. In our examples, we will always present P with any zero rows and columns removed. $\square$

Remark 2.4. There are several conditions on the univariate mask R at play in the surrounding discussion. Let $n \geq 2$, and let τ be the output of the coset sum with input R in n dimensions. Consider the following statements:

(i) R is interpolatory (or equivalently, τ is interpolatory)

(ii) R satisfies the univariate sub-QMF condition,

(iii) $f(R;\cdot)$ has an sos representation,

(iv) τ satisfies the sub-QMF condition,

(v) $f(\tau;\cdot)$ has an sos representation,

(vi) Condition ($\diamondsuit$) holds.

By the Fejér-Riesz Lemma, (ii) and (iii) are equivalent. (vi) implies (v) by Theorem 2.2, and (v) clearly implies (iv). Under (i), (ii)/(iii) implies (v) by Theorem 2.1, and (iv) implies (ii)/(iii) by Equation (2.4), so (ii), (iii), (iv), and (v) are all equivalent in this case. (vi) is strictly stronger than (ii)/(iii), even under (i), as seen in Example 2.2 in the next section (see Remark 2.6). If $n = 2$, then (iv) and (v) are equivalent by Theorem 2.4 of [11]. For $n \geq 3$, it is unknown if (iv) implies (v), but by Theorem 2.5 of [11], (iv) does not imply (v) if τ is a general multidimensional lowpass mask (i.e., not coming from the coset sum). $\square$

The following result provides a partial converse to Theorem 2.2.

Proposition 2.1. *Let R, H, τ, n, and f be as in Theorem 2.2, with H not necessarily satisfying Condition ($\diamondsuit$), and let x and P be as in its proof so that $f = x^*Px$. If P is weakly diagonally dominant (hence positive semidefinite), then H satisfies Condition ($\diamondsuit$).*

Proof. Suppose $Q \in M_K(\mathbb{R})$ is a square, Hermitian, weakly diagonally dominant matrix with nonnegative diagonal entries, for K some positive integer, such that $e^*Qe = 0$, for

e the column vector of all ones with length K. Then $0 = \sum_{i,j=1}^{K} Q_{i,j} \geq \sum_{i=1}^{K}(Q_{i,i} - \sum_{j\neq i}|Q_{i,j}|) \geq 0$, by the weak diagonal dominance of Q. Moreover, each of the summands $Q_{i,i} - \sum_{j\neq i}|Q_{i,j}| \geq 0$, so this equality forces $Q_{i,i} = \sum_{j\neq i}|Q_{i,j}|$ for all $1 \leq i \leq K$. Since the first inequality must be an equality, rearranging gives $\sum_{i=1}^{K}\sum_{j\neq i}(Q_{i,j} + |Q_{i,j}|) = 0$, and since each of the summands is nonnegative, $Q_{i,j} = -|Q_{i,j}|$ for all $1 \leq i, j \leq K$, $i \neq j$.

In the present case, the equality $f = x^*Px$ is clear by inspection of this product. Since $0 = f(0) = e^*Pe$ from the positive accuracy condition, the conditions on P imply that we may apply the above result to P, which gives us Condition ($\diamondsuit$). $\square$

The next corollary of Theorem 2.2 shows some simple sufficient conditions for Condition ($\diamondsuit$) to hold, hence for the associated lowpass mask to give rise to a tight wavelet filter bank. The fact that these conditions are not necessary can be seen easily, for example, by observing that many filters in Example 2.4 in the next section do not satisfy the conditions in the corollary but satisfy Condition ($\diamondsuit$). It should be noted that under these conditions, the filter coefficients $h(k)$ of τ are nonnegative for all $k \in \mathbb{Z}^n$, a case which has also been studied in [13] without the coset sum structure on the lowpass filter.

Corollary 2.1. *Let R be a positive accuracy mask with lowpass filter H, such that $R(0) = 1$. Suppose that $H(k) \geq 0$ for all $k \in \mathbb{Z}$, and $H(0) \geq (2^n - 2)/(2^n - 1)$, for some integer $n \geq 2$. Then H satisfies Condition ($\diamondsuit$) for this n, and a tight wavelet filter bank exists with $\tau = \mathcal{C}_n[R]$ as the lowpass mask.*

Proof. By Theorem 2.2, it suffices to show that the filter H satisfies Condition ($\diamondsuit$). Clearly, $H(2k)H(2j) \geq 0$ for all $j, k \neq 0$, and it remains to check that $\alpha(k) \geq 0$ for each k. We observe that

$$2^{n-1}\alpha(k) = ((2^n - 1)H(0) - (2^n - 2))(H(2k) + H(-2k)) + \sum_{\substack{j\in\mathbb{Z} \\ j\neq -2k,0}} H(j)H(j+2k),$$

and clearly the last sum and $H(2k)+H(-2k)$ are nonnegative, so $\alpha(k) \geq 0$ if $(2^n - 1)H(0) \geq 2^n - 2$, which is the stated condition on $H(0)$. $\square$

2.2 Examples

We illustrate our findings in the previous section by presenting some examples in this section. The first example is a tight wavelet filter bank constructed from the coset sum lowpass filter with the input B-spline of order 2.

Example 2.1. [From Interpolatory B-spline Filter of Order 2] For the (centered) B-spline of order 2 (also called the centered hat function), we consider the interpolatory mask $R(\omega) = 2^{-1}(1+\cos(\omega))$, $\omega \in \mathbb{T}$. Then, for $n \geq 2$, the n-dimensional coset sum lowpass mask is $\tau(\omega) = 2^{-n}(1 + \sum_{\nu\in\Gamma'} \cos(\nu \cdot \omega))$, $\omega \in \mathbb{T}^n$. From Equation (2.4) in the proof of Theorem 2.1, we see that

$$f(\tau;\omega) = \frac{2}{2^n}\sum_{\nu\in\Gamma'} f(R;\nu\cdot\omega) = \sum_{\nu\in\Gamma'} \frac{1}{2^{n+1}}(1-\cos(2\nu\cdot\omega)) = \sum_{\nu\in\Gamma'} \left|\frac{1}{2^{\frac{n}{2}+1}}(1-e^{-2i\nu\cdot\omega})\right|^2,$$

which gives us an sos representation for f, where the simple identity $2(1-\cos\omega) = |1-e^{-i\omega}|^2$, $\omega \in \mathbb{T}$ (which may be seen as a simple application of the Fejér-Riesz Lemma), is used for the last equality.

Since the univariate filter H associated with the mask R satisfies Condition ($\diamondsuit$) for all dimensions $n \geq 2$ (in fact, for any interpolatory filter H, Condition ($\diamondsuit$) is independent of n, hence holds true for all $n \geq 2$ if it holds true for any specific n), by Theorem 2.2, we have a matrix factorization of $f = x^*Px$ with a positive semidefinite matrix $P \in M_{2^n}(\mathbb{R})$ of the form

$$P = \frac{1}{2^{n+2}}\begin{bmatrix} 1 & & & & -1 \\ & 1 & & & -1 \\ & & \ddots & & \vdots \\ & & & 1 & -1 \\ -1 & -1 & \cdots & -1 & 2^n-1 \end{bmatrix}$$

and $x^* = [(e^{2i\nu\cdot\omega})_{\nu\in\Gamma'}, 1]$, where Remark 2.3 is used for the reduction of the matrix. The

Cholesky decomposition of P as $P = LL^*$ is given with the lower triangular matrix

$$L = \frac{1}{2^{1+n/2}} \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ -1 & -1 & \cdots & -1 & 0 \end{bmatrix}$$

and in this case, L^*x is exactly the same as the sos representation we obtained above using the approach of Theorem 2.1.

Either way, we get an sos decomposition of $f(\tau;\omega)$ with $2^n - 1$ generators. Thus, by Result 1.11, we obtain a tight wavelet frame with τ with $2^{n+1} - 1$ wavelet masks, which we index by $\eta \in \Gamma'$ for the first $2^n - 1$, and $\nu \in \Gamma$ for the last 2^n:

$$q_{1,\eta}(\omega) = -\frac{1}{2^{3n/2+1}}(1 - e^{2i\eta\cdot\omega}) \left(1 + \sum_{\nu\in\Gamma'} \cos(\nu \cdot \omega)\right).$$

$$q_{2,\nu}(\omega) = \frac{e^{i\nu\cdot\omega}}{2^{n/2}} - \frac{1}{2^{3n/2+1}}(1 + e^{2i\nu\cdot\omega}) \left(1 + \sum_{\mu\in\Gamma'} \cos(\mu \cdot \omega)\right),$$

Note that $q_{1,\eta}$ have 1 vanishing moment and the $q_{2,\nu}$ have 2 vanishing moments, so the tight wavelet frame has 1 vanishing moment.

Figure 2.1 depicts[1] the filters in dimension 2. In this case we have an sos decomposition of f with 3 generators, so we get 7 wavelet filters. In [31, Example 2.6], another construction is given which also yields a tight frame with 7 wavelet filters, which have smaller support, but decreased directionality and lack of symmetry. In [42, Example 5.2], it is shown that 2 sos generators (hence 6 wavelet filters) are actually sufficient, and they arrive at a very similar filter bank to ours, the main difference being that our $q_{2,(0,1)}$ and $q_{2,(1,1)}$ are essentially combined in their construction to yield one filter with larger support and loss of symmetry. In [12, Example 4.7], the authors construct a tight wavelet frame with this same lowpass filter, but with only 5 wavelet filters, which have smaller support, but decreased

[1] In the diagrams for filters in this paper, the bold-faced number is used to represent the value of the filter at the origin.

$$\begin{array}{ccc} & 1/2 & 1/2 \\ 1/2 & \mathbf{1} & 1/2 \\ 1/2 & 1/2 & \end{array}$$

(a) $\tau(\omega)$

$$\begin{array}{ccc} & -1/4 & -1/4 \\ -1/4 & \mathbf{3/2} & -1/4 \\ -1/4 & -1/4 & \end{array}$$

(b) $q_{2,(0,0)}(\omega)$

$$\begin{array}{ccccc} & 1/8 & 1/8 & -1/8 & -1/8 \\ 1/8 & 1/4 & 0 & \mathbf{-1/4} & -1/8 \\ 1/8 & 1/8 & -1/8 & -1/8 & \end{array}$$

(c) $q_{1,(1,0)}(\omega)$

$$\begin{array}{ccccc} & -1/8 & -1/8 & -1/8 & -1/8 \\ -1/8 & -1/4 & 7/4 & \mathbf{-1/4} & -1/8 \\ -1/8 & -1/8 & -1/8 & -1/8 & \end{array}$$

(d) $q_{2,(1,0)}(\omega)$

$$\begin{array}{ccc} & -1/8 & -1/8 \\ -1/8 & \mathbf{-1/4} & -1/8 \\ -1/8 & 0 & 1/8 \\ 1/8 & 1/4 & 1/8 \\ 1/8 & 1/8 & \end{array}$$

(e) $q_{1,(0,1)}(\omega)$

$$\begin{array}{ccc} & -1/8 & -1/8 \\ -1/8 & \mathbf{-1/4} & -1/8 \\ -1/8 & 7/4 & -1/8 \\ -1/8 & -1/4 & -1/8 \\ -1/8 & -1/8 & \end{array}$$

(f) $q_{2,(0,1)}$

$$\begin{array}{ccccc} & & & -1/8 & -1/8 \\ & & -1/8 & \mathbf{-1/4} & -1/8 \\ & 1/8 & 0 & -1/8 & \\ 1/8 & 1/4 & 1/8 & & \\ 1/8 & 1/8 & & & \end{array}$$

(g) $q_{1,(1,1)}(\omega)$

$$\begin{array}{ccccc} & & & -1/8 & -1/8 \\ & & -1/8 & \mathbf{-1/4} & -1/8 \\ & -1/8 & 7/4 & -1/8 & \\ -1/8 & -1/4 & -1/8 & & \\ -1/8 & -1/8 & & & \end{array}$$

(h) $q_{2,(1,1)}(\omega)$

Figure 2.1: Wavelet filters and lowpass filter from the B-spline of order 2 in Example 2.1 ($n = 2$).

directionality and no symmetry. $\square$

Remark 2.5. It is easy to see that the filters in Figure 2.1 all have symmetry. In fact, if R is a symmetric filter, i.e., $H(k) = H(-k)$ for all $k \in \mathbb{Z}$, then the output of the coset sum τ will have symmetry through the origin (among other symmetries), so it is a natural question whether or not it is possible to obtain highpass filters with this property. It is not difficult to show that the highpass masks $q_{2,\nu}$ in Result 1.11 will have symmetry under this condition, and that the highpass masks $q_{1,j}$ in that result will be symmetric precisely when the sos generators g_j have symmetry. For the sos representations constructed in Theorem 2.1, this requires $f(R;\omega)$ to have a representation $|p(2\omega)|^2$ for symmetric p. For the sos representations constructed in Theorem 2.2, after constructing P and x as in its proof, this symmetry requires a decomposition of $P = AA^*$ with the property that $A_j^* x$ is symmetric for each $1 \leq j \leq J$, where A_j is the jth column of the matrix A (see discussion preceding Remark 2.1). The conditions under which these representations and decompositions exist require further investigation. □

Example 2.2. [From Interpolatory Deslauriers-Dubuc Filters] For the Deslauriers-Dubuc (DD) filters of order $2k$ [17, 18, 22], $k \geq 1$, we have

$$R(\omega) = \cos^{2k}(\omega/2) P_k(\sin^2(\omega/2)),$$

where

$$P_k(x) = \sum_{j=0}^{k-1} \binom{k-1+j}{j} x^j.$$

When $k = 1$, $R(\omega)$ is the B-spline of order 2 mask we discussed already in Example 2.1. These filters are interpolatory and have positive accuracy, and as proved in [37], these masks satisfy the univariate sub-QMF condition for each $k \geq 1$, and thus by Theorem 2.1, for each $k \geq 1$ and dimension $n \geq 2$, there is an sos representation for $f(\tau; \cdot)$, where τ is the output of the coset sum with dimension n and input DD mask R of order $2k$.

Since $R_0(\omega) = 1/\sqrt{2}$ for the DD mask R of order $2k$, we see that

$$f(R; \omega) = 1/2 - 2\left(R(\omega) - 1/2\right)^2 = 2R(\omega)(1 - R(\omega)),$$

where the fact that $0 \leq R(\omega) \leq 1$ (also proved in [37]) ensures that both factors are nonnegative. Since $f(R; \omega/2)$ is a univariate nonnegative trigonometric polynomial, by the Fejér-Riesz Lemma, there exists a trigonometric polynomial p such that $f(R; \omega) = |p(2\omega)|^2$. Since P_k is the unique polynomial of degree $k-1$ satisfying $(1-y)^k P_k(y) + y^k P_k(1-y) = 1$ for all $y \in [0, 1]$ (see [14]), we see that $1 - R(\omega) = (1 - \cos^{2k}(\omega/2) P_k(\sin^2(\omega/2))) = \sin^{2k}(\omega/2) P_k(\cos^2(\omega/2))$. Hence $f(R; \omega)$ has a factor of $\sin^{2k}(\omega/2)$, and as a result, $p(\omega)$ has a root of order k at 0. This in turn implies that the sos generators for $f(\tau; \omega)$, with the coset sum generated τ from the DD mask R of order $2k$, have a root of order k at 0, so the highpass masks $q_{1,j}$ in Result 1.11 have at least k vanishing moments. Since τ satisfies the interpolatory property, its flatness number is at least its accuracy number, which is $2k$. Then by Proposition 1.1, the masks $q_{2,\nu}, \nu \in \Gamma$ have at least $2k$ vanishing moments, which means that all of the wavelet masks in the tight wavelet frame constructed from Result 1.11 have at least k vanishing moments.

For the DD mask of order 4 (i.e. $k = 2$), $R(\omega) = 2^{-1}(1 + (9/8)\cos(\omega) - (1/8)\cos(3\omega))$, and for τ the output of the coset sum in dimension $n \geq 2$, we have

$$f(\tau;\omega) = \frac{1}{2^{n+7}} \sum_{\nu\in\Gamma'} (46 - 63\cos(2\nu\cdot\omega) + 18\cos(4\nu\cdot\omega) - \cos(6\nu\cdot\omega)).$$

Thus, if p is a trigonometric polynomial such that

$$2^{n+7}|p(\omega)|^2 = 46 - 63\cos(\omega) + 18\cos(2\omega) - \cos(3\omega),$$

we have $f(\tau;\omega) = \sum_{\nu\in\Gamma'} |p(2\nu\cdot\omega)|^2$. One possible such a choice of p is given by

$$2^{(n+7)/2}p(\omega) = \sqrt{\frac{7}{2} - 2\sqrt{3}}(7 + 4\sqrt{3} - e^{-i\omega})(1 - e^{-i\omega})^2.$$

As we can see, here $p(\omega)$ has a double root at $\omega = 0$, which, together with the argument above for the vanishing moments of $q_{2,\nu}$, implies that the wavelet masks in the tight wavelet frame with lowpass mask τ constructed by Result 1.11 all have at least 2 vanishing moments.

□

Remark 2.6. We note that while the DD mask of order 4 satisfies the assumptions of Theorem 2.1, it does not satisfy Condition (♢), so it does not satisfy the hypotheses of Theorem 2.2. Indeed, since the interpolatory condition implies that $H(0) = 1$, we get

$$\alpha(2) = \frac{1}{2^{n-1}} \sum_{j\in\mathbb{Z}} H(j)H(j+4) = \frac{1}{2^{n-1}}(H(-3)H(1) + H(-1)H(3)) = \frac{2}{2^{n-1}}\left(\frac{-9}{256}\right) < 0.$$

Example 2.3. [From B-spline Filter of Order 3] We see that the (centered) B-spline of order 3, with $R(\omega) = 2^{-3}(e^{i\omega} + 3 + 3e^{-i\omega} + e^{-i2\omega})$, satisfies Condition (♢) only for $n = 2$ and 3. Thus, for example when $n = 2$, by Theorem 2.2 we have the representation of f as x^*Px with

$$P = \frac{1}{64}\begin{bmatrix} 6 & -1 & -1 & -4 \\ -1 & 6 & -1 & -4 \\ -1 & -1 & 6 & -4 \\ -4 & -4 & -4 & 12 \end{bmatrix}, \quad x = \begin{bmatrix} e^{-2i\omega_1} \\ e^{-2i\omega_2} \\ e^{-2i(\omega_1+\omega_2)} \\ 1 \end{bmatrix}.$$

Finding a Cholesky factorization for P gives $P = LL^*$ with the following L:

$$L = \frac{1}{8}\begin{bmatrix} \sqrt{6} & 0 & 0 & 0 \\ -1/\sqrt{6} & \sqrt{35/6} & 0 & 0 \\ -1/\sqrt{6} & -\sqrt{7/30} & 2\sqrt{7/5} & 0 \\ -4/\sqrt{6} & -4\sqrt{7/30} & -2\sqrt{7/5} & 0 \end{bmatrix},$$

which corresponds to an sos representation of f with 3 sos generators:

$$\begin{aligned} 64f(\omega) &= (1/6)|4 + e^{-2i(\omega_1+\omega_2)} + e^{-2i\omega_2} - 6e^{-2i\omega_1}|^2 \\ &\quad + (7/30)|4 + e^{-2i(\omega_1+\omega_2)} - 5e^{-2i\omega_2}|^2 + (28/5)|1 - e^{-2i(\omega_1+\omega_2)}|^2. \end{aligned}$$

Alternatively, using the method described in Remark 2.1, we obtain the following representation of f as x^*Px:

$$P = \frac{1}{64}\begin{bmatrix} 5 & 0 & 0 & 0 & -5 \\ 0 & 5 & 0 & 0 & -5 \\ 0 & 0 & 4 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ -5 & -5 & -4 & -1 & 15 \end{bmatrix}, \quad x = \begin{bmatrix} e^{-2i\omega_1} \\ e^{-2i\omega_2} \\ e^{-2i(\omega_1+\omega_2)} \\ e^{-2i(\omega_1-\omega_2)} \\ 1 \end{bmatrix}.$$

This corresponds to a representation of f as

$$64f(\omega) = 5|1 - e^{-2i\omega_1}|^2 + 5|1 - e^{-2i\omega_2}|^2 + 4|1 - e^{-2i(\omega_1+\omega_2)}|^2 + |1 - e^{-2i(\omega_1-\omega_2)}|^2,$$

an sos representation of f with 4 sos generators. Observe that each of these sos generators only has 2 nonzero coefficients, which corresponds to the Cholesky factor of P only having nonzeroes on its main diagonal and last row (and in fact, this property of the Cholesky factor holds generally when using the method in Remark 2.1, as can be seen by inspecting the product $P = LL^*$). In this case, we see that the naive approach leads to 1 additional sos generator (and wavelet mask) with each of these sos generators having only 2 complex exponentials, while the former approach has slightly fewer sos generators (and thus wavelet masks) with the generators being more complicated. $\square$

Example 2.4. [From Burt-Adelson Filters] We consider the parametrized family of lowpass filters from [7], which with our normalization is $R(\omega) = 2^{-1}(a + \cos(\omega) + (1 - a)\cos(2\omega))$, for $a \in \mathbb{R}$. We refer to these as the Burt-Adelson (BA) masks with parameter a, and the associated filters as the BA filters with parameter a. A picture of the coset sum generated lowpass filter from the BA filter in two dimensions may be seen in Figure 2.2. It is easy to see that most components of Condition ($\diamondsuit$) hold automatically for these filters, with the only nontrivial one being $\alpha(1) \geq 0$. This can be shown to be equivalent to the condition

$$|a - (2^{n+1} - 3)/(2^{n+1} - 2)| \leq 2^{n/2}/(2^{n+1} - 2) \tag{2.7}$$

where a is the parameter of the BA filters. Thus, for each fixed $n \geq 2$, if we choose the parameter a to be $a(n) := (2^{n+1} - 3)/(2^{n+1} - 2)$, then the BA filter with parameter $a(n)$ satisfies Condition ($\diamondsuit$) for this same n. We observe that $a(n)$ is an increasing function of n with limit 1, and the BA filter with parameter 1 corresponds to nothing but the B-spline of order 2 studied in Example 2.1. Thus, the scaling function with parameter $a(n)$ looks Gaussian (though compactly supported) for small n, and approaches the piecewise linear B-spline as n gets larger (see also the diagrams in [7], though the parameter a used there is $a/2$ with our notation).

Let $n \geq 2$ be fixed, and suppose that the parameter a of the BA filters satisfies the condition in (2.7). We let

$$\begin{aligned}
v &= 2^{-n}[0, (2^n - 1)a^2 - (2^{n+1} - 3)a + 2^n - 9/4],\\
b &= -2^{-n}(2^n - 1)((2^n - 1)a^2 - (2^{n+1} - 3)a + 2^n - 9/4),\\
C &= \frac{-(1-a)^2}{4 \cdot 2^n} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\\
B &= \frac{1}{4 \cdot 2^n} \begin{bmatrix} (2^{n+1} - 3)(1-a)^2 & -(1-a)^2 \\ -(1-a)^2 & -(2^{n+1} - 1)a^2 + 2(2^{n+1} - 3)a - 2(2^n - 3) \end{bmatrix}.
\end{aligned}$$

Then with the block B appearing $2^n - 1$ times on the diagonal, we have that P is given as

$$
\begin{array}{ccccc}
(1-a)/2 & 1/2 & \boldsymbol{a} & 1/2 & (1-a)/2
\end{array}
$$

$$
\Big\downarrow \ \mathcal{C}_2 \text{ (Coset Sum)}
$$

$$
\begin{array}{ccccc}
 & & (1-a)/2 & & (1-a)/2 \\
 & & 1/2 & 1/2 & \\
(1-a)/2 & 1/2 & \boldsymbol{3a-2} & 1/2 & (1-a)/2 \\
 & 1/2 & 1/2 & & \\
(1-a)/2 & & (1-a)/2 & &
\end{array}
$$

Figure 2.2: Lowpass filter from the Burt-Adelson filter in Example 2.4 ($n = 2$).

in Equation (2.6). □

2.3 Summary

In this chapter, we combined the idea of constructing multivariate tight wavelet frames from sums of squares representations of the function $f(\tau;\cdot)$ with that of the coset sum method, which generates a nonseparable multivariate lowpass mask τ from a univariate lowpass mask R. As one example of the fruitfulness of combining these ideas, consider Example 2.2, where we find that for any number of vanishing moments l and dimension $n \geq 2$, there is a tight wavelet frame with all highpass masks having l vanishing moments for a coset sum generated nonseparable interpolatory lowpass mask, namely the output of the coset sum method for the input Deslauriers-Dubuc mask of order $2l$.

We demonstrated a variety of methods for obtaining sos representations for nonnegative trigonometric polynomials, and showed how the structured support of the filters we were considering could be used to reduce the number of sos generators for the associated trigonometric polynomials. In fact, in some cases, the structure of the support can be used to prove the existence of a sum of squares representation, while other conditions may fail, as we saw in Remark 2.6: Here we used the interpolatory condition, which in light of Equations (1.9) and (1.10) might be interpreted as a condition on the support of a filter, to prove the existence of a sum of squares representation for $f(\tau;\cdot)$. On the other hand, Condition ($\diamondsuit$), which does not make use of this interpolatory structure, fails for this example. Further exploration of the cases in which information of this kind may be leveraged to find sos representations for nonnegative trigonometric polynomials is an interesting open problem suggested by these findings.

In the next chapter, we generalize the results of this chapter to the case of prime dilation factor, focusing on the case of interpolatory lowpass masks. We will see that when $p > 2$, even the interpolatory case becomes much more complicated, and will require much more careful study of the lattice $\mathbb{Z}^n/p\mathbb{Z}^n$ in order to find the desired sums of squares representation. We will again strongly leverage the structured support of $f(\tau;\cdot)$ to prove that this representation exists.

Chapter 3

Prime Coset Sum Lowpass Masks

3.1 Introduction

In this chapter, we generalize the results of Chapter 2 to the case of prime dilation. Many of the results are more complicated in this section, so while it is possible to generalize the matrix-based methods of Section 2.1.4 for finding sos representations to this setting, we focus on the case of interpolatory lowpass masks, which is more conceptual.

It is worth noting that nonseparable multivariate wavelet construction with prime dilation has not been discussed much in the literature, unless it is included as a special case of constructions for general dilation matrices. In this setting, even following the construction of Result 1.11, which works for any dilation matrix, requires finding sos representations for highly complicated trigonometric polynomials, and the approximation of the smoothness, vanishing moments, and other wavelet desiderata is similarly complicated. As such, it may be difficult to know where to begin if one wishes to construct a nonseparable multivariate tight wavelet frame with certain properties outside the setting of dyadic dilation. The Prime Coset Sum Method (PCS) [39] allows one to construct a multidimensional lowpass mask from a one-dimensional lowpass mask with prime dilation, so that the resulting lowpass mask is nonseparable, and the output mask has the same prime dilation as the input. Furthermore, a certain minimum accuracy number of the output mask is guaranteed by the method, with this bound related to similar properties of the input mask (see Section 1.4.2). In this chapter, we will prove some new results about PCS lowpass masks (see Sections 3.2.1 and 3.2.2). More precisely, we will improve the understanding of PCS by proving new bounds on the accuracy and flatness numbers of the output lowpass mask, which guarantee that under certain conditions, the accuracy number of the input lowpass mask is exactly the accuracy number of the output lowpass mask. We will also prove that when the input mask is interpolatory and satisfies the sub-QMF condition, the output also satisfies these properties, and the associated refinable function belongs to $L^2(\mathbb{R}^n)$, using a result from [30].

When R is interpolatory and satisfies the sub-QMF condition, which we call a PCSTF-admissible mask, we will be able to prove the existence of an sos representation for $f(\tau;\cdot)$ as in Result 1.11, for τ arising from the PCS method. We then use the generators of this sos representation to construct highpass masks satisfying the UEP conditions with τ, which leads to a tight wavelet frame for $L^2(\mathbb{R}^n)$. We refer to this new method as the *Prime Coset Sum Method for Tight Wavelet Frames* (PCSTF), and use our results on the vanishing moments of the highpass masks coming from Result 1.11 to prove that a lower bound for the number of vanishing moments of the generated frames is proportional to the accuracy number of the input lowpass mask. These results are informed by the new bounds on the accuracy and flatness numbers of lowpass masks arising from PCS.

In Section 3.2, we show that PCS preserves the sub-QMF condition for interpolatory filters. We then prove new results on the flatness and accuracy numbers of lowpass masks generated from PCS. In Section 3.3, we define a particular group action, and use this to obtain an orbit decomposition of the set of coset representatives input to PCS. We use this orbit decomposition to prove that the desired sos representations for $f(\tau;\cdot)$ exist for PCSTF-admissible lowpass masks input to the PCS method, which gives us the Prime Coset Sum Method for Tight Wavelet Frames (PCSTF) when combined with Result 1.11. We also prove bounds on the vanishing moments of PCSTF-generated tight wavelet frames. In Section 3.3.4, we give two examples of our full method, and concluding remarks are given in Section 3.4.

3.2 New Properties of PCS Lowpass Masks

In this section, we prove a handful of new properties about PCS. We start by proving that PCS preserves the sub-QMF condition for interpolatory lowpass masks, before proving some new results about the accuracy and flatness numbers of PCS-generated lowpass masks.

Let us fix some notation which will be used throughout the sequel. Let p be a fixed odd prime, and let I be a set of distinct coset representatives of $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ including 0. Let $n \geq 2$ be the spatial dimension, and let Γ be a set of distinct coset representatives of $\mathbb{Z}^n/p\mathbb{Z}^n$ including 0. We use the notation Γ', I' to denote the corresponding sets of nonzero cosets, and we denote by $(a \pmod p)$ the number $b \in I$ such that $a \equiv b \pmod p$, for any integer a. In some cases, we will have multiple sets I under consideration, so we will use the notation $(a \,(\mathrm{mod}\, p : I))$ when this clarification is necessary. The notation $(a^{-1} \pmod p)$ refers to the multiplicative inverse of a in $\mathbb{Z}_p$, when $a \not\equiv 0 \pmod p$, and we will adopt the convention that $((a^{-1} \pmod p)b \pmod p)$ will be abbreviated as $(a^{-1}b \pmod p)$ throughout.

Throughout this chapter, we consider $\mathcal{M} = pI$, where I is the $n \times n$ identity matrix, and choose $\Gamma^* = (2\pi/p)\{0, \ldots, p-1\}^n$.

3.2.1 PCS Lowpass Masks and the Sub-QMF Condition

Let R be a univariate lowpass mask with dilation p, and let τ be the output of PCS with input R in n dimensions, and with a fixed set Γ. In presenting our results on PCS lowpass masks below and throughout the paper, we will use sets $\mathbf{M}(\nu), \nu \in \Gamma'$ defined as

$$\mathbf{M}(\nu) = \{(\nu', j) \in \Gamma' \times I' : j\nu' \equiv \nu \pmod{p\mathbb{Z}^n}\}. \tag{3.1}$$

We will say more about these sets after defining a particular group action in Section 4.1 (see Remark after Lemma 3.3), but for now the only property we require of them is that $|\mathbf{M}(\nu)| = p - 1$. To see why this is true, suppose that for $j \in I'$, $\nu_1, \nu_2 \in \Gamma'$, $j\nu_1 \equiv j\nu_2 \equiv \nu \pmod{p\mathbb{Z}^n}$. Then $j(\nu_1 - \nu_2) \equiv 0 \pmod{p\mathbb{Z}^n}$, and since j is invertible mod p, $\nu_1 \equiv \nu_2 \pmod{p\mathbb{Z}^n}$, which means that $\nu_1 = \nu_2$, since Γ' is a set of distinct coset representatives. Then if $(\nu_1, j) \in \mathbf{M}(\nu)$ for some $\nu_1 \in \Gamma', j \in I'$, no other pairs $(\nu_2, j), \nu_2 \neq \nu_1$ appear in

$\mathbf{M}(\nu)$. Moreover, Γ' has coset representatives for each nonzero element of $\mathbb{Z}^n/p\mathbb{Z}^n$, so letting $\nu_1 \equiv (j^{-1} \pmod{p})\nu \pmod{p\mathbb{Z}^n}$, we get that $j\nu_1 \equiv j(j^{-1} \pmod{p})\nu \equiv \nu \pmod{p\mathbb{Z}^n}$, so for each $j \in I'$, there is an element $(\nu_1, j) \in \mathbf{M}(\nu)$, which proves that $|\mathbf{M}(\nu)| = |I'| = p-1$.

Lemma 3.1 (Polyphase components of τ)**.** *The polyphase components of τ are given as follows:*

$$\tau_0(\omega) = \frac{p}{(p-1)p^{n/2}} \left(1 - p^{n-1} + \frac{1}{\sqrt{p}} \sum_{\nu \in \Gamma'} R_0(\omega \cdot \nu)\right),$$

$$\tau_\nu(\omega) = \frac{\sqrt{p}}{(p-1)p^{n/2}} \sum_{(\nu',j)\in \mathbf{M}(\nu)} R_j(\omega \cdot \nu') \exp\left(i\omega \cdot \frac{j\nu' - \nu}{p}\right), \quad \nu \in \Gamma'.$$

Proof. We start by computing, to try to write $\tau(\omega) = \sum_{\nu\in\Gamma} g_\nu(p\omega)\exp(i\omega\cdot\nu)$ as in (1.6), in which case the functions $p^{n/2}g_\nu$ are the polyphase components, where the uniqueness of these comes from the reconstruction formula in Equation (1.7). Starting from the PCS definition (1.14), and using (1.6) for R, where $R_j, j \in I$ are its polyphase components, we see that

$$\begin{aligned}
\tau(\omega) &= \frac{1}{(p-1)p^{n-1}} \left(1 - p^{n-1} + \frac{1}{\sqrt{p}} \sum_{\nu\in\Gamma'} \sum_{j\in I} R_j(p\omega\cdot\nu)\exp(ij\omega\cdot\nu)\right) \\
&= \frac{1}{(p-1)p^{n-1}} \left(1 - p^{n-1} + \frac{1}{\sqrt{p}} \sum_{\nu\in\Gamma'} R_0(p\omega\cdot\nu)\right) \\
&\quad + \frac{\sqrt{p}}{(p-1)p^n} \sum_{\nu\in\Gamma'} \sum_{j\in I'} R_j(p\omega\cdot\nu)\exp(ij\omega\cdot\nu).
\end{aligned}$$

We observe that the first line of the last expression above gives us the desired formula for τ_0. Since we are summing over all $\nu \in \Gamma'$ and $j \in I'$ in the last line, and each coset in Γ' is congruent to $j\nu \pmod{p\mathbb{Z}^n}$ for exactly $p-1$ pairs (ν, j), we repurpose ν for this product and sum over the pairs $(\nu', j) \in \Gamma' \times I'$ with $j\nu' \equiv \nu$, which leads to the following formula for $\tau(\omega) - p^{-n/2}\tau_0(p\omega)$:

$$\begin{aligned}
&\frac{\sqrt{p}}{(p-1)p^n} \sum_{\nu\in\Gamma'} \sum_{(\nu',j)\in\mathbf{M}(\nu)} R_j(p\omega\cdot\nu')\exp(i\omega\cdot j\nu') \\
&= \frac{\sqrt{p}}{(p-1)p^n} \sum_{\nu\in\Gamma'} \left(\sum_{(\nu',j)\in\mathbf{M}(\nu)} R_j(p\omega\cdot\nu')\exp(i\omega\cdot(j\nu'-\nu))\right)\exp(i\omega\cdot\nu).
\end{aligned}$$

For each pair $(\nu', j) \in \mathbf{M}(\nu)$, $j\nu' \equiv \nu \,(\mathrm{mod}\, p\mathbb{Z}^n)$ by definition, so the expression inside the large parentheses here is indeed a function of $p\omega$. This completes the proof. $\square$

We next find an upper bound for $|\tau_\nu(\omega)|^2$ when $\nu \in \Gamma'$.

Lemma 3.2 (Squared polyphase components of τ)**.** *For $\nu \in \Gamma'$, the polyphase components of τ satisfy*

$$|\tau_\nu(\omega)|^2 \leq \frac{p}{(p-1)p^n} \sum_{(\nu',j)\in\mathbf{M}(\nu)} |R_j(\omega \cdot \nu')|^2.$$

Proof. By viewing the formula for $\tau_\nu(\omega)$ given in Lemma 3.1 as, up to a multiplicative constant, the inner product of the vectors $[R_j(\omega \cdot \nu')]_{(\nu',j)\in\mathbf{M}(\nu)}$ and $x = [\exp(i\omega \cdot (\nu - j\nu')/p)]_{(\nu',j)\in\mathbf{M}(\nu)}$ for some ordering of the set $\mathbf{M}(\nu)$, we may apply the Cauchy-Schwarz Inequality to conclude that

$$|\tau_\nu(\omega)|^2 \leq \frac{p}{(p-1)^2 p^n} \left[(p-1) \sum_{(\nu',j)\in\mathbf{M}(\nu)} |R_j(\omega \cdot \nu')|^2 \right],$$

where we have used the fact that $|\mathbf{M}(\nu)| = p-1$ to see that $\|x\|_2^2 = p-1$. $\square$

Definition 3.1. We say that a univariate lowpass mask R satisfying the interpolatory and sub-QMF conditions is *PCSTF-admissible.*

Note that every PCSTF-admissible mask R has positive accuracy, since R is necessarily lowpass and satisfies the sub-QMF condition (c.f. comment after Equation (1.5)).

Theorem 3.1 (PCS preserves the sub-QMF condition for interpolatory masks)**.** *Let R be PCSTF-admissible. Then τ satisfies the multivariate sub-QMF condition.*

Proof. Using the fact that the PCS method preserves the interpolatory and positive accuracy properties (c.f. Result 1.5(i-ii)), and applying Equation (1.10), we have that $\tau_0(\omega) =$

$p^{-n/2}$. From (1.18) and Lemma 3.2, we have that

$$\begin{aligned} f(\tau;\omega) &= \frac{p^n-1}{p^n} - \sum_{\nu\in\Gamma'} |\tau_\nu(p\omega)|^2 \\ &\geq \frac{p^n-1}{p^n} - \frac{p}{(p-1)p^n} \sum_{\nu\in\Gamma'} \sum_{(\nu',j)\in\mathbf{M}(\nu)} |R_j(p\omega\cdot\nu')|^2 \\ &= \frac{p^n-1}{p^n} - \frac{p}{(p-1)p^n} \sum_{\nu\in\Gamma'} \sum_{j\in I'} |R_j(p\omega\cdot\nu)|^2 \\ &= \frac{p}{(p-1)p^n} \sum_{\nu\in\Gamma'} \left(\frac{p-1}{p} - \sum_{j\in I'} |R_j(p\omega\cdot\nu)|^2 \right) \\ &= \frac{p}{(p-1)p^n} \sum_{\nu\in\Gamma'} f(R;\omega\cdot\nu) \geq 0. \end{aligned} \tag{3.2}$$

This completes the proof. □

Comparing with the case $p = 2$ as in Equation (2.4), the corresponding set $\mathbf{M}(\nu)$ there has size 1 for each $\nu \in \Gamma'$. This means there is no need for the Cauchy-Schwarz Inequality, so the inequality after (3.2) is an equality in this case, which gives an sos representation immediately. In the case $p > 2$, we will need to study $f(\tau;\cdot)$ much more carefully in order to obtain an sos representation.

Applying Result 1.3, we immediately obtain the following corollary:

Corollary 3.1. *Let R be PCSTF-admissible. Then the refinable function associated with τ is a compactly supported $L^2(\mathbb{R}^n)$ function.*

3.2.2 Flatness and Accuracy Numbers of PCS Lowpass Masks

When τ is the output of PCS with input R, Result 1.5(ii) tells us that its accuracy number is at least the minimum of the flatness and accuracy numbers of R. In light of Theorem 1.1, we see the importance of the flatness and accuracy numbers of τ for the vanishing moments of highpass masks constructed from Result 1.11, so in the next result, we investigate the relationship between the flatness numbers of τ and R.

Proposition 3.1 (Flatness number of τ)**.** *Let R have flatness number s, and let t be the smallest even integer such that $D^t(1-R)(0) \neq 0$. Then $1 \leq s \leq t$ and the flatness number* *of τ lies between s and t (inclusive of s, t). Furthermore,*

(i) If $R(\omega) = R(-\omega), \omega \in \mathbb{T}$, then the flatness number of τ is $s = t$.

(ii) If $\Gamma = -\Gamma$, then the flatness number of τ is t.

Proof. Let α be a multiindex with $|\alpha| > 0$. Then from (1.14):

$$D^{\alpha}\tau(\omega) = \frac{1}{(p-1)p^{n-1}} \sum_{\nu \in \Gamma'} \nu^{\alpha} D^{|\alpha|} R(\omega \cdot \nu),$$

so

$$D^{\alpha}\tau(0) = \frac{D^{|\alpha|}R(0)}{(p-1)p^{n-1}} \sum_{\nu \in \Gamma'} \nu^{\alpha}.$$

This proves the lower bound immediately. If we consider $\alpha = [t, 0, \ldots, 0]^T$, $\sum_{\nu \in \Gamma'} \nu^{\alpha} > 0$, so we see that $D^{\alpha}\tau(0) \neq 0$. Furthermore,

(i) If $R(\omega) = R(-\omega)$, then $R(\omega) = c_0 + \sum_{k=1}^{d} c_k \cos(k\omega)$, for some c_k and d, so for any integer $j \geq 0$, $D^{2j+1}R(\omega) = (-1)^{j+1} \sum_{k=1}^{d} c_k k^{2j+1} \sin(k\omega)$, which is 0 at $\omega = 0$. Then in this case, $s = t$.

(ii) If $\Gamma = -\Gamma$, then for any multiindex α with odd $|\alpha|$, $\sum_{\nu \in \Gamma'} \nu^{\alpha} = \sum_{\nu \in \Gamma'} (-\nu)^{\alpha} = -\sum_{\nu \in \Gamma'} \nu^{\alpha}$, which means this sum is 0. Then $D^{\alpha}\tau(0) = 0$ for all α with odd $|\alpha|$, which completes the proof. $\square$

The following corollary illustrates a simple but illuminating use of this proposition.

Corollary 3.2 (Accuracy and flatness numbers equal and even)**.** *Let R be interpolatory, and have accuracy and flatness numbers both equal to an even integer $m > 0$. Then τ has accuracy and flatness numbers both equal to m.*

Proof. By Result 1.5(i-ii), τ is interpolatory, and its accuracy number is at least m. By the interpolatory property, the flatness number of τ is at least its accuracy number, but the proposition above tells us this flatness number is m ($= s = t$, in the notation of the proposition). Thus both numbers are equal to m. $\square$

Since a PCSTF-admissible mask is interpolatory, Corollary 3.2 says that in particular τ has accuracy and flatness numbers both equal to an even positive m whenever the univariate mask R is PCSTF-admissible with the same property.

Outside the setting of this corollary, the corresponding result for accuracy numbers is more complicated. For a multiindex α, $\gamma \in \Gamma^* \setminus \{0\}$, and $k \in I = \{0, \ldots, p-1\}$, let

$$V_{\gamma,\alpha}(k) = \sum_{\substack{\nu \in \Gamma' \\ \gamma\cdot\nu \equiv (2\pi k)/p}} \nu^\alpha, \tag{3.3}$$

where $\gamma \cdot \nu \equiv (2\pi k)/p$ is taken $\bmod 2\pi\mathbb{Z}$. Let $A_{\gamma,\alpha} = [V_{\gamma,\alpha}(i^{-1}j \pmod p)]_{i,j=1}^{p-1}$. $V_{\gamma,\alpha}$ clearly depends on the set Γ, but we suppress this from the notation since the set Γ should always be clear from context.

We now present our result about the relationship between the accuracy numbers of τ and R.

Proposition 3.2 (Accuracy number of τ and $A_{\gamma,\alpha}$)**.** *Let R be a univariate lowpass mask with positive accuracy, and let τ be the output of PCS in n dimensions. For any positive integer m, let $\rho_m = [D^m R(2\pi j/p)]_{j=1}^{p-1}$, and let $e = [1]_{j=1}^{p-1}$. Then the accuracy number of τ is the least integer a for which there is some multiindex α, $|\alpha| = a$, and $\gamma \in \Gamma^* \setminus \{0\}$ such that*

$$A_{\gamma,\alpha}\rho_a \neq -D^\alpha R(0) V_{\gamma,\alpha}(0) e.$$

Proof. Let α be a multiindex with $|\alpha| > 0$, and let $\gamma \in \Gamma^* \setminus \{0\}$. Then

$$\begin{aligned} D^\alpha \tau(\gamma) &= \frac{1}{(p-1)p^{n-1}} \sum_{\nu \in \Gamma'} \nu^\alpha D^{|\alpha|} R(\gamma \cdot \nu) \\ &= \frac{1}{(p-1)p^{n-1}} \sum_{j=0}^{p-1} V_{\gamma,\alpha}(j) D^{|\alpha|} R\left(\frac{2\pi j}{p}\right) \\ &= \frac{1}{(p-1)p^{n-1}} (V_{\gamma,\alpha}(0) D^{|\alpha|} R(0) + (A_{\gamma,\alpha}\rho_{|\alpha|})(1)). \end{aligned}$$

Observe that when $1 \leq i \leq p-1$,

$$\gamma \cdot \nu \equiv \frac{2\pi(i^{-1}j \pmod p)}{p} \pmod{2\pi\mathbb{Z}} \Leftrightarrow i\gamma \cdot \nu \equiv \frac{2\pi j}{p} \pmod{2\pi\mathbb{Z}}.$$

Then we see that $D^\alpha \tau(i\gamma) = 0$, $1 \leq i \leq p-1$, if and only if $A_{\gamma,\alpha}\rho_{|\alpha|} = -D^{|\alpha|} R(0) V_{\gamma,\alpha}(0) e$.

$\square$

In fact, from the last sentence of the proof, we see that we only need to check the condition of the proposition for some collection $M \subseteq \Gamma^*$ such that $\gamma_1 \not\equiv i\gamma_2 \,(\mathrm{mod}\, 2\pi\mathbb{Z}^n)$ for any $\gamma_1 \neq \gamma_2 \in M$, $i \in I'$.

In the following corollaries, we suppose the same setting as in Proposition 3.2. In particular, we use a to denote the accuracy number of τ.

Corollary 3.3. *(i) (Special case of γ and α) Let $m \geq 1$, and suppose*

$$A_{(2\pi/p)e_1, me_1}\rho_m \neq -D^m R(0) V_{(2\pi/p)e_1, me_1}(0)e.$$

Then $a \leq m$.

(ii) (Gap between flatness and accuracy numbers) If the flatness number of R is strictly larger than its accuracy number $m \geq 1$, and $A_{(2\pi/p)e_1, me_1}$ is nonsingular, then $a = m$.

(iii) (Zero-normalization for Γ) Suppose Γ is such that for all $\nu \in \Gamma$, if $\nu(i) \equiv 0 \,(\mathrm{mod}\, p)$, then $\nu(i) = 0$, $1 \leq i \leq n$. If the accuracy number of R is m and $A_{(2\pi/p)e_1, me_1}$ is nonsingular, then $a \leq m$. When R is interpolatory, $a = m$.

Proof. (i) Immediate from Proposition 3.2 for $\gamma = (2\pi/p)e_1$ and $\alpha = me_1$.

(ii) Observe that $D^m R(0) = 0$, and since $\rho_m \neq 0$ and $A_{(2\pi/p)e_1, me_1}$ is nonsingular, $A_{(2\pi/p)e_1, me_1}\rho_m \neq 0$, so by (i), $a \leq m$. By Result 1.5(ii), $a \geq m$, and therefore $a = m$.

(iii) Under the assumptions on Γ, $V_{(2\pi/p)e_1, me_1}(0) = 0$. Since $\rho_m \neq 0$ (from the assumption on the accuracy number of R) and $A_{(2\pi/p)e_1, me_1}$ is assumed to be nonsingular, $A_{(2\pi/p)e_1, me_1}\rho_m \neq 0$, so $a \leq m$. When R is interpolatory, its flatness number is at least m, and Result 1.5(ii) yields $a \geq m$, which proves the equality.

□

These corollaries encourage further study of the matrices $A_{\gamma,\alpha}$, particularly to find conditions under which these are invertible. We carry this out in the next section.

3.2.3 Studying the Matrices $A_{\gamma,\alpha}$

In the following section, we will study the matrices $A_{\gamma,\alpha}$, and in particular, we will prove formulas for their eigenvectors and eigenvalues. We begin with some examples of $A_{\gamma,\alpha}$.

For the first few choices of p (i.e. for $p = 3$ and 5), the matrices $A_{\gamma,\alpha}$ are:

$$\begin{bmatrix} V_{\gamma,\alpha}(1) & V_{\gamma,\alpha}(2) \\ V_{\gamma,\alpha}(2) & V_{\gamma,\alpha}(1) \end{bmatrix}, \quad \begin{bmatrix} V_{\gamma,\alpha}(1) & V_{\gamma,\alpha}(2) & V_{\gamma,\alpha}(3) & V_{\gamma,\alpha}(4) \\ V_{\gamma,\alpha}(3) & V_{\gamma,\alpha}(1) & V_{\gamma,\alpha}(4) & V_{\gamma,\alpha}(2) \\ V_{\gamma,\alpha}(2) & V_{\gamma,\alpha}(4) & V_{\gamma,\alpha}(1) & V_{\gamma,\alpha}(3) \\ V_{\gamma,\alpha}(4) & V_{\gamma,\alpha}(3) & V_{\gamma,\alpha}(2) & V_{\gamma,\alpha}(1) \end{bmatrix},$$

where $V_{\gamma,\alpha}$ is defined as in (3.3) for some set Γ of distinct coset representatives of $\mathbb{Z}^n/p\mathbb{Z}^n$ including 0, $\gamma \in \Gamma^* \setminus \{0\}$, and multiindex α in each case.

When $V_{\gamma,\alpha}(k) = k$, the leading principal minor of $A_{\gamma,\alpha}$ of order $(p-1)/2$ is called the *Maillet determinant* and has been studied extensively in the literature (e.g., in [9, 10, 45], and [55]). In particular, it is known that, for each odd prime p, the Maillet determinant does not vanish.

We begin by stating some easy properties of the matrices $A_{\gamma,\alpha}$. We then constructively show that $A_{\gamma,\alpha}$ is permutation similar to a circulant matrix, and use this to give an explicit formula for their eigenvalues and eigenvectors, which follows from the corresponding formulas for circulant matrices.

We will use J to denote the appropriately sized reversal matrix where it appears, so that if this is k, then the ith column of J is e_{k+1-i}, $1 \le i \le k$.

It is easy to see that $A_{\gamma,\alpha}$ is a Latin square matrix [40], with all diagonal entries equal to $V_{\gamma,\alpha}(1)$ (since $(i^{-1}i \,(\mathrm{mod}\, p)) = 1$), and all antidiagonal entries equal to $V_{\gamma,\alpha}(p-1)$ (since $((p-i)^{-1}i \,(\mathrm{mod}\, p)) = (p - i^{-1}i \,(\mathrm{mod}\, p)) = p-1$).

The matrix $A_{\gamma,\alpha}$ is *centrosymmetric* for every $\gamma \in \Gamma^* \setminus \{0\}$ and multiindex α. To see why this is so, let us first recall (c.f. [2]) that a matrix is called centrosymmetric if it is symmetric about its center, i.e., for even k, if a $k \times k$ matrix A satisfies $A(i,j) = A(k+1-i, k+1-j)$ for $1 \le i, j \le k$. Now let us show that the matrix $A_{\gamma,\alpha}$ is centrosymmetric.

Since $((p-i)^{-1} \pmod p) = (p - i^{-1} \pmod p)$, for $1 \le i, j \le p-1$,

$$\begin{aligned} A_{\gamma,\alpha}(i,j) &= V_{\gamma,\alpha}(i^{-1}j \pmod p) = V_{\gamma,\alpha}((p-i^{-1})(p-j) \pmod p) \\ &= V_{\gamma,\alpha}((p-i)^{-1}(p-j) \pmod p) = A_{\gamma,\alpha}(p-i, p-j), \end{aligned}$$

which proves that $A_{\gamma,\alpha}$ is centrosymmetric.

We recall that for an odd prime p, a nonzero element h of $\mathbb{Z}_p$ is called *primitive* if its powers generate the multiplicative group $(\mathbb{Z}_p)^\times$, i.e., if $\{(h^k \pmod p) : 1 \le k \le p-1\} = \{1, 2, \ldots, p-1\}$. It is well known (e.g., [43]) that a primitive always exists in $\mathbb{Z}_p$, and, in fact, that there are Euler's totient function, $\phi(p-1)$, many primitive elements of $\mathbb{Z}_p$. Our next theorem says that $A_{\gamma,\alpha}$ is similar to a circulant matrix via a permutation matrix, which is defined using a primitive of $\mathbb{Z}_p$.

Proposition 3.3 ($A_{\gamma,\alpha}$ is permutation similar to a circulant matrix)**.** *Let h be a primitive of $\mathbb{Z}_p$. Then $A_{\gamma,\alpha}$ is permutation similar to a circulant matrix whose first row is* $[V_{\gamma,\alpha}(1), V_{\gamma,\alpha}(h \pmod p), V_{\gamma,\alpha}(h^2 \pmod p), \ldots, V_{\gamma,\alpha}(h^{p-2} \pmod p)]$.

Proof. Let P be the permutation matrix that sends $e_{(h^j \pmod p)}$ to e_j for each $1 \le j \le p-1$. Explicitly, $P(i,j) = 1$ if $j = (h^i \pmod p)$, and is 0 otherwise. Then

$$\begin{aligned} (PA_{\gamma,\alpha}P^T)(i,j) &= \sum_{k=1}^{p-1} (PA_{\gamma,\alpha})(i,k) P^T(k,j) \\ &= \sum_{k=1}^{p-1} A_{\gamma,\alpha}(h^i \pmod p, k) P(j,k) \\ &= A_{\gamma,\alpha}(h^i \pmod p, h^j \pmod p) \\ &= V_{\gamma,\alpha}(h^{j-i} \pmod p), \end{aligned}$$

which implies that the matrix $PA_{\gamma,\alpha}P^T$ is Toeplitz. Since $h^{p-1-i} \equiv h^{-i} \equiv h^{1-(i+1)} \pmod p$, $(PA_{\gamma,\alpha}P^T)(i, p-1) = (PA_{\gamma,\alpha}P^T)(i+1, 1)$ for each $1 \le i \le p-2$, which proves that $PA_{\gamma,\alpha}P^T$ is in fact a circulant matrix. □

We illustrate this result using an example.

Example 3.1. Let us consider the case when $p = 5$. Then there are exactly two primitives, namely 2 and 3, in $\mathbb{Z}_5$. Following the arguments in the proof of Theorem 3.3, we see that the permutation matrices for $h = 2$ and $h = 3$ are

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{3.4}$$

respectively, and the circulant matrices corresponding to these are those with the first rows $v_2 = [V_{\gamma,\alpha}(1), V_{\gamma,\alpha}(2), V_{\gamma,\alpha}(4), V_{\gamma,\alpha}(3)]$, and $v_3 = [V_{\gamma,\alpha}(1), V_{\gamma,\alpha}(3), V_{\gamma,\alpha}(4), V_{\gamma,\alpha}(2)]$, respectively. In other words, $A_{5,m}$ is permutation similar to $\mathrm{Circ}(v_2)$ via the first permutation matrix in (3.4) and is permutation similar to $\mathrm{Circ}(v_3)$ via the second permutation matrix in (3.4), where $\mathrm{Circ}(\xi^T)$ is defined as the circulant matrix which has ξ^T as its first row. ■

We now list some corollaries of Theorem 3.3. Let us start with an immediate one.

Corollary 3.4. *$A_{\gamma,\alpha}$ is normal.*

Proof. Normality is equivalent to unitary diagonalizability [34], and in [41], it is shown that circulant matrices satisfy the latter condition. Since normality is preserved under conjugation by a unitary matrix [34], the matrices $A_{\gamma,\alpha}$ are also normal. □

We recall that circulant matrices commute because any $k \times k$ circulant matrix, with $k \geq 2$, is a polynomial in the forward shift permutation matrix $\mathrm{Circ}(e_2^T)$, the powers of which form a cyclic group of order k. In light of this and Proposition 3.3, it comes as no surprise that the matrices $A_{\gamma,\alpha}$ are actually polynomials in a single permutation matrix:

Corollary 3.5 ($A_{\gamma,\alpha}$ is a polynomial in a permutation matrix)**.** *Let h be a primitive of $\mathbb{Z}_p$, and let Q be the permutation matrix of order $p-1$, such that $Q(i,j) = 1$ if $A_{\gamma,\alpha}(i,j) = V_{\gamma,\alpha}(h)$, and is 0 otherwise. Then*

$$A_{\gamma,\alpha} = \sum_{k=1}^{p-1} V_{\gamma,\alpha}(h^k \pmod p) Q^k.$$

Proof. Clearly, Q^k is a permutation matrix for each $k \geq 1$, so suppose that for some $k-1 \geq 1$, $Q^{k-1}(i,j) = 1$ only when $A_{\gamma,\alpha}(i,j) = V_{\gamma,\alpha}(h^{k-1} \pmod p)$. This is the assumption when $k-1 = 1$, which handles the base case of the induction, so now we show that the corresponding equality holds for Q^k.

Since $Q(\ell,j) = 1$ if and only if $(\ell^{-1}j \pmod p) = h$, so $\ell = (h^{-1}j \pmod p)$, and we see that

$$\begin{aligned} Q^k(i,j) &= \sum_{\ell=1}^{p-1} Q^{k-1}(i,\ell)Q(\ell,j) \\ &= Q^{k-1}(i,(h^{-1}j \pmod p)), \end{aligned}$$

which is equal to 1 if and only if $A_{\gamma,\alpha}(i,(h^{-1}j \pmod p)) = V_{\gamma,\alpha}(h^{k-1} \pmod p)$. But we see that $i^{-1}(h^{-1}j) \equiv h^{k-1} \pmod p$ if and only if $i^{-1}j \equiv h^k \pmod p$, which completes the induction and the proof. □

Since $A_{\gamma,\alpha}$ is permutation similar to a circulant matrix (c.f. Proposition 3.3), and the eigenvalues and eigenvectors of circulant matrices are well understood (see, for example, [41]), we can write down the eigenvalues and eigenvectors of $A_{\gamma,\alpha}$ explicitly.

Theorem 3.2 (Eigensystem of $A_{\gamma,\alpha}$). *Let h be a primitive of $\mathbb{Z}_p$. For $\ell = 1,\ldots,p-1$, let $z_\ell := \exp(2\pi i\ell/(p-1))$ so that $\{z_\ell\}_{\ell=1}^{p-1}$ are the distinct $(p-1)$st roots of unity. Then if we define*

$$\xi_\ell = \sum_{k=1}^{p-1} z_\ell^k e_{(h^k \pmod p)}, \quad \ell = 1,\ldots,p-1,$$

the set $\{(p-1)^{-1/2}\xi_\ell\}_{\ell=1}^{p-1}$ is an orthonormal basis for $\mathbb{C}^{p-1}$, and for each $1 \leq \ell \leq p-1$, $A_{\gamma,\alpha}\xi_\ell = \lambda_{\gamma,\alpha;\ell}\xi_\ell$, where the $\lambda_{\gamma,\alpha;\ell}$ are given by:

$$\lambda_{\gamma,\alpha;\ell} = \sum_{k=1}^{p-1} z_\ell^k V_{\gamma,\alpha}(h^k \pmod p).$$

Proof. Letting P be the permutation matrix associated with the primitive h from the proof of Proposition 3.3, $PA_{\gamma,\alpha}P^T(j,k) = V_{\gamma,\alpha}(h^{k-j}\,(\mathrm{mod}\,p))$. Since

$$\sum_{k=1}^{p-1} V_{\gamma,\alpha}(h^{k-j}\,(\mathrm{mod}\,p))z_\ell^k = \sum_{k=1}^{p-1} V_{\gamma,\alpha}(h^k\,(\mathrm{mod}\,p))z_\ell^{k+j} = z_\ell^j \sum_{k=1}^{p-1} V_{\gamma,\alpha}(h^k\,(\mathrm{mod}\,p))z_\ell^k,$$

we see that $((PA_{\gamma,\alpha}P^T)[z_\ell^k]_{k=1}^{p-1})(j) = z_\ell^j \lambda_{\gamma,\alpha;\ell}$, so $([z_\ell^k]_{k=1}^{p-1}, \lambda_{\gamma,\alpha;\ell})$ is an eigenpair for $PA_{\gamma,\alpha}P^T$. Since $P^T e_k = e_{(h^k\,(\mathrm{mod}\,p))}$, we see that ξ_ℓ is an eigenvector for $A_{\gamma,\alpha}$ with eigenvalue $\lambda_{\gamma,\alpha;\ell}$ for all $1 \le \ell \le p-1$. The matrix $X = (p-1)^{-1/2}[z_\ell^k]_{k,\ell=1}^{p-1}$ is orthonormal [41], so $P^T X = [\xi_1\,\xi_2\,\cdots\,\xi_{p-1}]$ is also.

□

The eigenvectors and eigenvalues of $A_{\gamma,\alpha}$ found in Theorem 3.2 have some interesting properties. For example:

Corollary 3.6. *Let ξ_ℓ be the eigenvectors found in Theorem 3.2. For even ℓ, $J\xi_\ell = \xi_\ell$ holds, and for odd ℓ, $J\xi_\ell = -\xi_\ell$ holds.*

Proof. For a primitive h of $\mathbb{Z}_p$, there is some k, $1 \le k \le p-2$ such that $h^k \equiv p-1\,(\mathrm{mod}\,p)$, by definition of a primitive, since $h^{p-1} \equiv 1\,(\mathrm{mod}\,p)$. Then $h^{2k} \equiv 1\,(\mathrm{mod}\,p)$, which means that $2k = c(p-1)$ for an integer c. Given the range of k considered, this forces $c = 1$, and $k = (p-1)/2$. This gives the equality, for $1 \le k \le (p-1)/2$,

$$(h^k\,(\mathrm{mod}\,p)) + (h^{(p-1)/2+k}\,(\mathrm{mod}\,p)) = (h^k\,(\mathrm{mod}\,p)) + (p - h^k\,(\mathrm{mod}\,p)) = p.$$

Using this, we see that

$$\begin{aligned} J\xi_\ell &= \sum_{k=1}^{p-1} z_\ell^k e_{p-(h^k\,(\mathrm{mod}\,p))} = \sum_{k=1}^{p-1} z_\ell^k e_{(h^{(p-1)/2+k}\,(\mathrm{mod}\,p))} \\ &= \sum_{k=1}^{p-1} z_\ell^{k-(p-1)/2} e_{(h^k\,(\mathrm{mod}\,p))} = z_\ell^{-(p-1)/2}\xi_\ell. \end{aligned}$$

The result now follows since

$$z_\ell^{-(p-1)/2} = \exp\left(\frac{2\pi i\ell}{p-1}\left(-\frac{p-1}{2}\right)\right) = \exp(-\pi i\ell) = (-1)^\ell.$$

□

Now we relate the information about the eigenvectors ξ_ℓ of $A_{\gamma,\alpha}$ back to the question of accuracy numbers for PCS-generated lowpass masks τ.

Corollary 3.7 (ρ_m is nonconstant)**.** *Let τ be the output of PCS with input R, and let its accuracy number be a. Suppose R has accuracy number m, and ρ_m is not a constant vector (i.e., $\rho_m \neq ce$ for some $c \in \mathbb{C}$). If $A_{\gamma,\alpha}$ is nonsingular for some α with $|\alpha| = m$ and $\gamma \in \Gamma^* \setminus \{0\}$, then $a \leq m$. When R is interpolatory, $a = m$.*

Proof. We may write ρ_m as $\sum_{k=1}^{p-1} \alpha_k \xi_k$, for some $\alpha_k \in \mathbb{C}$, and since ρ_m is nonconstant, some $\alpha_{k^*} \neq 0$ for $k^* \neq p-1$. But $A_{\gamma,\alpha}\rho_m = \sum_{k=1}^{p-1} \alpha_k \lambda_{\gamma,\alpha;k}\xi_k$, and since all of the $\lambda_{\gamma,\alpha;k} \neq 0$, ξ_{k^*} must have a nonzero coefficient in this representation. But this means that $A_{\gamma,\alpha}\rho_m \neq ce$ for any $c \in \mathbb{C}$, so the condition of Proposition 3.2 is satisfied for some α with $|\alpha| = m$, and the accuracy number of τ is at most m. The last statement is proved as in Corollary 3.3(iii). □

It is easy to see that if the set Γ used in PCS is symmetric, i.e., $\Gamma = -\Gamma$, then the output τ of the method will be the same if R is used as the input, or if $\frac{1}{2}(R + R(-\cdot))$ is used. Indeed, from Equation (1.14),

$$\begin{aligned}
\mathcal{C}_{n,p}[R] &= \frac{1}{(p-1)p^{n-1}}\left(1 - p^{n-1} + \sum_{\nu\in\Gamma'} R(\omega\cdot\nu)\right) \\
&= \frac{1}{(p-1)p^{n-1}}\left(1 - p^{n-1} + \sum_{\substack{\nu\in\Gamma' \\ \nu>_{\mathrm{lex}}0}} R(\omega\cdot\nu) + R(\omega\cdot(-\nu))\right) \\
&= \frac{1}{(p-1)p^{n-1}}\left(1 - p^{n-1} + \sum_{\nu\in\Gamma'} \frac{1}{2}(R(\omega\cdot\nu) + R(-\omega\cdot\nu))\right) \\
&= \mathcal{C}_{n,p}[(R + R(-\cdot))/2].
\end{aligned}$$

Moreover, replacing R with its symmetrized version may only increase its flatness and accuracy numbers. If R has accuracy number m, and $a \leq m-1$, $D^a(\frac{1}{2}(R + R(-\cdot)))(\gamma) = \frac{1}{2}(D^a R(\gamma) + (-1)^a D^a R(-\gamma))$, so this is necessarily zero for $\gamma \in \Gamma^* \setminus \{0\}$, and the argument for the flatness number is similar. For these reasons, inputs R satisfying $R = R(-\cdot)$ are an interesting special case.

Proposition 3.4 (Symmetric Γ result in singular $A_{\gamma,\alpha}$)**.** *Suppose $\Gamma = -\Gamma$. Then $A_{\gamma,\alpha}$ is singular, its nullity is at least $(p-1)/2$, and $V_{\gamma,\alpha}(0) = 0$ whenever $|\alpha|$ is odd.*

Let $(A_{\gamma,\alpha})_1$ be the leading principle submatrix of $A_{\gamma,\alpha}$ of order $(p-1)/2$. Then $(A_{\gamma,\alpha})_1$ has eigensystem $\{((\xi_\ell)_1, \frac{1}{2}\lambda_{\gamma,\alpha;\ell}) : 1 \le \ell \le p-1, \ell \equiv |\alpha| \,(\mathrm{mod}\, 2)\}$, where $(x)_1 := [x(k)]_{k=1}^{(p-1)/2}$. Using the notation of Proposition 3.2, if $R = R(-\cdot)$, the accuracy number of τ is the least integer a for which there is some multiindex α, $|\alpha| = a$, and $\gamma \in \Gamma^* \setminus \{0\}$ such that

$$(A_{\gamma,\alpha})_1 (\rho_a)_1 \neq -\frac{1}{2} D^a R(0) V_{\gamma,\alpha}(0)(e)_1.$$

Proof. We observe that

$$V_{\gamma,\alpha}(k) = \sum_{\substack{\nu \in \Gamma' \\ \gamma\cdot\nu \equiv \frac{2\pi k}{p}}} \nu^\alpha = \sum_{\substack{\nu \in \Gamma' \\ \gamma\cdot\nu \equiv \frac{2\pi(p-k)}{p}}} (-1)^{|\alpha|}\nu^\alpha = (-1)^{|\alpha|} V_{\gamma,\alpha}(p-k \,(\mathrm{mod}\, p)).$$

When $k=0$, this says that $V_{\gamma,\alpha}(0) = (-1)^{|\alpha|}V_{\gamma,\alpha}(0)$, and when $k \neq 0$, this gives a dependence relation between columns of $A_{\gamma,\alpha}$, since $V_{\gamma,\alpha}(i^{-1}j \,(\mathrm{mod}\, p)) = (-1)^{|\alpha|}V_{\gamma,\alpha}(i^{-1}(p-j)\,(\mathrm{mod}\, p))$. Counting these dependence relations gives us the statement about the nullity of $A_{\gamma,\alpha}$.

Observe that Corollary 3.6 gives $J\xi_\ell = (-1)^\ell \xi_\ell$, and the above shows that $A_{\gamma,\alpha}J = (-1)^{|\alpha|}A_{\gamma,\alpha}$. Let $(A_{\gamma,\alpha})_1'$ denote the first $(p-1)/2$ columns of $A_{\gamma,\alpha}$. Then if $\ell \equiv |\alpha| \,(\mathrm{mod}\, 2)$, we have

$$\lambda_{\gamma,\alpha;\ell}\xi_\ell = A_{\gamma,\alpha}\xi_\ell = (A_{\gamma,\alpha})_1'(\xi_\ell)_1 + (-1)^{\ell+|\alpha|}(A_{\gamma,\alpha})_1' J^2 (\xi_\ell)_1 = 2(A_{\gamma,\alpha})_1'(\xi_\ell)_1,$$

using $J^2 = I$. Reading off the first $(p-1)/2$ rows of the previous equation, we obtain the formula for the eigensystem of $(A_{\gamma,\alpha})_1$.

To obtain the last result, we observe that $JA_{\gamma,\alpha} = (-1)^{|\alpha|}A_{\gamma,\alpha}$, since $(p-i)^{-1}j \equiv p - i^{-1}j \,(\mathrm{mod}\, p)$, which means that $A_{\gamma,\alpha}(p-i,j) = (-1)^{|\alpha|}A_{\gamma,\alpha}(i,j)$, in light of the relation $V_{\gamma,\alpha}(p-k) = (-1)^{|\alpha|}V_{\gamma,\alpha}(k)$ shown above. Restricting to the first $(p-1)/2$ columns, this also clearly holds for $(A_{\gamma,\alpha})'_1$. Now since $(D^a R)(-\omega) = (-1)^a D^a[R(-\cdot)](\omega)$, we have that

$$\begin{aligned}
J\rho_a &= [D^a R(2\pi(p-k)/p)]_{k=1}^{p-1} \\
&= (-1)^a [D^a[R(-\cdot)](2\pi k/p)]_{k=1}^{p-1} \\
&= (-1)^a [D^a R(2\pi k/p)]_{k=1}^{p-1} = (-1)^a \rho_a,
\end{aligned}$$

using $R = R(-\cdot)$ in the last line. Then similarly to the case for ξ_ℓ above, we see that

$$A_{\gamma,\alpha}\rho_a = (A_{\gamma,\alpha})'_1(\rho_a)_1 + (-1)^{a+|\alpha|}(A_{\gamma,\alpha})'_1 J^2(\rho_a)_1 = 2(A_{\gamma,\alpha})'_1(\rho_a)_1,$$

so the negation of the condition of Proposition 3.2, $A_{\gamma,\alpha}\rho_a = -D^a R(0)V_{\gamma,\alpha}(0)e$, holds if and only if $(A_{\gamma,\alpha})'_1(\rho_a)_1 = -\frac{1}{2}D^a R(0)V_{\gamma,\alpha}(0)e$. Left-multiplying both sides of the latter equality by J, the effect on the left hand side is multiplication by $(-1)^{|\alpha|}$, and the right hand side is unchanged, since $Je = e$. But when $a = |\alpha|$ is odd, then by symmetry, $D^a R(0) = 0$, so for either parity of $|\alpha|$, we may multiply the right hand side by $(-1)^{|\alpha|}$ without change. This means the last $(p-1)/2$ rows in this equation are redundant, which gives the equivalence with the equation in the statement of the proposition. □

3.3 Prime Coset Sum Tight Wavelet Frames

We would like more detailed information about the function $f(\tau;\cdot)$ (c.f. (1.18)) when τ is the output of PCS with PCSTF-admissible input R in n dimensions, and with a fixed set Γ. In particular, we would like to know whether $f(\tau;\cdot)$ has an sos representation in this setting, and as it happens, this is guaranteed to exist for any p and n, and any set Γ. This fact clearly relies heavily on the structure of PCS, since in [11, Th. 2.5], it is shown that there exist lowpass masks in 3 dimensions for which $f(\tau;\cdot)$ has no sos representation, even when it is nonnegative. We begin by defining a particular group action (see Section 1.3.1 for more on these), which we will use to define the orbit decomposition of the set Γ', and will be useful for finding sos representations for $f(\tau;\cdot)$. After this, we prove a lemma from lattice theory which is used to ensure that variables with certain properties exist.

3.3.1 Group Action

In our setting, there is a natural action of the multiplicative group $(\mathbb{Z}_p)^\times$ on the set Γ'. Let I be a set of distinct coset representatives of $\mathbb{Z}_p$ containing 0, and let I' be the corresponding set of nonzero cosets. Then we may define $k \circ \nu$ as the element $\nu' \in \Gamma'$ such that $k\nu \equiv \nu' \pmod{p\mathbb{Z}^n}$, which is well-defined because Γ contains distinct coset representatives. This is also independent of the choice of I, since if $k \equiv j \pmod{p}$, then $k\nu \equiv j\nu \pmod{p\mathbb{Z}^n}$, which are then both congruent to the same element $\nu' \in \Gamma'$. We will refer to this as *the group action of* $(\mathbb{Z}_p)^\times$ *on* Γ'. In particular, given Γ, we can always find a set $M \subset \Gamma'$ of distinct orbit representatives for this action, so that $\Gamma' = \bigcup_{\mu \in M} \mathcal{O}_\mu$, where $\mathcal{O}_\mu = (\mathbb{Z}_p)^\times \circ \mu$ is the notation we will use below for the orbit of μ in this group action, for $\mu \in \Gamma'$.

We will use the following fact about the group action just described at several points in what follows.

Lemma 3.3 (All orbits have size $p-1$). *In the group action of* $(\mathbb{Z}_p)^\times$ *on* Γ'*, each orbit has* $p-1$ *elements.*

Proof. Let I be a set of distinct coset representatives of $\mathbb{Z}_p$ including 0. Given $\nu \in \Gamma'$, the map $\cdot \circ \nu : I' \to \Gamma'$ is injective, since for $\nu \in \Gamma'$, there is some $1 \le i \le n$ such that $\nu(i) \not\equiv 0 \pmod{p}$, and if $k\nu \equiv j\nu \pmod{p\mathbb{Z}^n}$, then $k\nu(i) \equiv j\nu(i) \pmod{p}$, so $k \equiv j \pmod{p}$, which means that $k = j$, since I contains distinct coset representatives. By definition, the orbit $\mathcal{O}_\nu$ is the image of this map, so $|\mathcal{O}_\nu| = |I'| = p-1$. □

Remark: For the set $\mathbf{M}(\nu)$ in (3.1), we observe that we could index $\mathbf{M}(\nu)$ by its first component, which just covers the elements of $\mathcal{O}_\nu$. Indeed, when $j\nu' \equiv \nu \pmod{p\mathbb{Z}^n}$, $\nu' \equiv (j^{-1} \pmod{p})\nu \pmod{p\mathbb{Z}^n}$, which shows that $\nu' \in \mathcal{O}_\nu$. Then $\mathbf{M}(\nu)$ could equivalently be written $\{(\nu', j) \in \mathcal{O}_\nu \times I' : j\nu' \equiv \nu \pmod{p\mathbb{Z}^n}\}$. We may also index $\mathbf{M}(\nu)$ by $j \in I'$, in which case $\mathbf{M}(\nu) = \{((j^{-1} \pmod{p}) \circ \nu, j) : j \in I'\}$. ■

Note that since $|\Gamma'| = p^n - 1$, and the size of each orbit in the group action of $(\mathbb{Z}_p)^\times$ on Γ' is $p-1$, M must have $(p^n-1)/(p-1) = \sum_{k=0}^{n-1} p^k$ elements. This suggests one method for finding M:

Lemma 3.4. *Let Γ be a set of distinct coset representatives for $\mathbb{Z}^n/p\mathbb{Z}^n$, and for $1 \leq k \leq n$, let*

$$M_k = \{\nu \in \Gamma' : \nu(j) \equiv 0 \pmod p, 1 \leq j \leq k-1, \nu(k) \equiv 1 \pmod p\}.$$

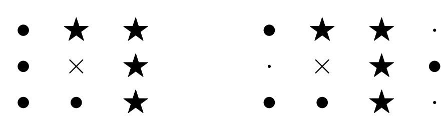
Then $M = \bigcup_{k=1}^{n} M_k$ is a complete set of distinct orbit representatives in the group action of $(\mathbb{Z}_p)^\times$ on Γ'.

Proof. Given $\nu \in \Gamma'$, there must be some k, $1 \leq k \leq n$, such that $\nu(j) \equiv 0 \pmod p$ for all $1 \leq j \leq k-1$, and $\nu(k) \not\equiv 0 \pmod p$. Then $\nu(k)$ is invertible mod p, and $\nu' = (\nu(k)^{-1} \pmod p) \circ \nu \in \Gamma'$ has $\nu'(j) \equiv \nu(k)^{-1}\nu(j) \equiv 0 \pmod p$ for $1 \leq j \leq k-1$, and $\nu'(k) \equiv \nu(k)^{-1}\nu(k) \equiv 1 \pmod p$, so $\nu' \in M_k$. Clearly, $\nu = \nu(k) \circ \nu'$, which shows that every orbit has a representative in M. Since $|M_k| = p^{n-k}$, $|M| = \sum_{k=1}^{n} |M_k| = \sum_{k=0}^{n-1} p^k = (p^n - 1)/(p-1)$, which means that the orbit representatives in M must be distinct. $\square$

Example 3.2. For $p = 3, n = 2$, let $\Gamma_1 = \{-1, 0, 1\}^2$ and $I = \{-1, 0, 1\}$. Using Lemma 3.4, we see that one choice of M is given by $\{(0,1), (1,-1), (1,0), (1,1)\}$, and for each $\mu \in M$, $\mathcal{O}_\mu = \{\mu, -\mu\}$.

For the same p, n and I, if we let $\Gamma_2 = (\{-1,0,1\}^2 \setminus \{-e_1\}) \cup \{2e_1\}$, the same M again gives a set of distinct orbit representatives in the group action of $(\mathbb{Z}_3)^\times$ on Γ'. When $\mu \in M \setminus \{e_1\}$, $\mathcal{O}_\mu = \{\mu, -\mu\}$, and when $\mu = e_1$, $\mathcal{O}_\mu = \{\mu, 2\mu\}$, so $(-1) \circ e_1 = 2e_1$.

A diagram of these sets for $\Gamma := \Gamma_1$ or Γ_2 is depicted in Figure 3.1, where the $\cdot$ indicate elements of $\mathbb{Z}^2$ that do not belong to Γ, $\times$ indicates the origin, and ★ indicate members of M. ■



(a) Γ_1 (b) Γ_2

Figure 3.1: Examples of Γ for $p = 3, n = 2$ from Example 3.2

3.3.2 A Lemma from Lattice Theory

In the following lemma, we show that a nonnegative trigonometric polynomial with nonzero coefficients only on a dimension-m subspace may be written coherently as a trigonometric polynomial in m variables $\omega \cdot \zeta_i$ for some $\zeta_i \in \mathbb{Z}^n$, $1 \le i \le m$. Our interest will be in the special cases of this lemma for $m = 1$ or 2, but we give the more general statement.

Lemma 3.5. *Let $\{x_1, \dots, x_m\} \subset \mathbb{Z}^n$ be a linearly independent set. Then there are vectors $\zeta_i \in \mathbb{Z}^n, 1 \le i \le m$, such that if $Z = [\zeta_1|\zeta_2|\cdots|\zeta_m]$, then*

(a) $\{x_1, \dots, x_m\} \subset Z\mathbb{Z}^m$, and

(b) $Z : \mathbb{Z}^m \to \mathbb{Z}^n$ is injective mod p, i.e., for $a \in \mathbb{Z}^m$,

$$\text{if } Za \equiv 0 \,(\operatorname{mod} p\mathbb{Z}^n), \text{ then } a \equiv 0 \,(\operatorname{mod} p\mathbb{Z}^m).$$

Proof. Let $\mathcal{L} = \operatorname{span}(\{x_1, \dots, x_m\}) \cap \mathbb{Z}^n$, which is an m-dimensional lattice. Then by [4, Th. 10.4], there are vectors $\zeta_i \in \mathbb{Z}^n, 1 \le i \le m$ such that each element of $\mathcal{L}$ may be represented uniquely as Za for some $a \in \mathbb{Z}^m$, where Z is the matrix with columns $\zeta_i, 1 \le i \le m$. This shows point (a) immediately.

For (b), if we let $u \in p\mathcal{L} = \operatorname{span}(\{x_1, \dots, x_m\}) \cap (p\mathbb{Z}^n)$, then $u/p \in \mathcal{L}$, so $u/p = Za$, or $u = Z(pa)$. But this means that $a' = pa$ is the unique integer vector such that $u = Za'$. This proves that if $Za \equiv 0 \,(\operatorname{mod} p\mathbb{Z}^n)$, then $a \equiv 0 \,(\operatorname{mod} p\mathbb{Z}^m)$. □

3.3.3 Prime Coset Sum Method for Tight Wavelet Frames (PCSTF)

We are now ready to present our main result of this chapter, a new method for constructing interpolatory tight wavelet frames with prime dilation for $L^2(\mathbb{R}^n)$ based on combining Result 1.11 with PCS generated lowpass masks.

We start by showing that an sos representation for $f(\tau;\cdot)$ (c.f. (1.18)) exists, provided that τ is generated by PCS from a PCSTF-admissible univariate mask R, and then investigate the vanishing moments of the highpass masks arising from Result 1.11 using this τ and sos representation.

Theorem 3.3. *Let R be PCSTF-admissible, and let τ be the output of PCS with input R in n dimensions. Then $f(\tau;\cdot)$ has an sos representation.*

The idea of the proof is as follows: We know that if $G(\omega), \omega \in \mathbb{R}^n$ is a nonnegative trigonometric polynomial in one or two variables $\omega \cdot \zeta$ or $\omega \cdot \zeta_i$, $i = 1, 2$, where $\zeta, \zeta_1, \zeta_2 \in \mathbb{Z}^n$, then $G(\omega) = |g(\omega \cdot \zeta)|^2$ in the first case, by the Fejér-Riesz Lemma (Result 1.8(a)), or else is a sum of finitely many squares $|g_j(\omega \cdot \zeta_1, \omega \cdot \zeta_2)|^2$, by Result 1.8(b). The goal is then to decompose $f(\tau;\cdot)$ into a sum of finitely many nonnegative trigonometric polynomials G_μ, such that for each G_μ we may find appropriate ζ, or $\zeta_i, i = 1, 2$, with the property that G_μ is a trigonometric polynomial in $\omega \cdot \zeta$ or $\omega \cdot \zeta_i, i = 1, 2$. Combining this with the aforementioned results will then guarantee the existence of an sos representation for $f(\tau;\cdot)$. Our main nonnegativity assumption is that R satisfies the sub-QMF condition, and this will serve as a guide in the proof, since we will try to decompose $f(\tau;\cdot)$ into G_μ which are lower bounded by $f(R;\omega \cdot \zeta)$ or some suitable combination of two $f(R;\omega \cdot \zeta_i), i = 1, 2$.

Proof. Let I be a set of distinct coset representatives of $\mathbb{Z}_p$ containing 0, and let I' be the corresponding set of nonzero cosets. Let Γ be the set of distinct cosets of $\mathbb{Z}^n/p\mathbb{Z}^n$ containing 0 used in the PCS method for constructing τ, and let Γ' be the set of nonzero cosets. In the group action of $(\mathbb{Z}_p)^\times$ on Γ' (see Section 3.3.1), which we recall is denoted $k \circ \nu$ for $k \in I'$ and $\nu \in \Gamma'$, let M be a set of distinct orbit representatives. We define the following vector, which will significantly simplify our calculations, where $k \in I'$, $\mu \in M$ and $\omega \in \mathbb{T}^n$:

$$\mathcal{R}_{k,\mu}(\omega) = \left[R_{(k^{-1}j \,(\mathrm{mod}\, p))}(\omega \cdot (k \circ \mu)) \exp\left(i\omega \cdot \frac{(k^{-1}j \,(\mathrm{mod}\, p))(k \circ \mu)}{p} \right) \right]_{j \in I'}.$$

Observe that

$$\begin{aligned} \frac{p-1}{p} - \|\mathcal{R}_{k,\mu}(p\omega)\|^2 &= \frac{p-1}{p} - \sum_{j \in I'} |R_{(k^{-1}j \,(\mathrm{mod}\, p))}(p\omega \cdot (k \circ \mu))|^2 \\ &= \frac{p-1}{p} - \sum_{j \in I'} |R_j(p\omega \cdot (k \circ \mu))|^2 \\ &= f(R; \omega \cdot (k \circ \mu)). \end{aligned}$$

In particular, this shows that $\|\mathcal{R}_{k,\mu}(\omega)\|^2 \le \frac{p-1}{p}$, $\forall \omega \in \mathbb{T}^n$, since $f(R;\cdot) \ge 0$.

Now we compute, from Equation (3.2),

$$\begin{aligned} f(\tau;\omega) &= \frac{p^n-1}{p^n} - \sum_{\nu\in\Gamma'} |\tau_\nu(p\omega)|^2 \\ &= \frac{1}{p^n}\sum_{\mu\in M}\left[p-1-p^n\sum_{j\in I'}|\tau_{(j\circ\mu)}(p\omega)|^2\right] =: \frac{1}{p^n}\sum_{\mu\in M} G_\mu(p\omega). \end{aligned}$$

Now using our computation of τ_ν, $\nu\in\Gamma'$ from Lemma 3.1 and the remark after Lemma 3.3 on the set $\mathbf{M}(\nu)$, we see that $G_\mu(\omega)$ equals

$$\begin{aligned} p-1-&\frac{p}{(p-1)^2}\sum_{j\in I'}\sum_{k,\ell\in I'} R_{(k^{-1}j\,(\mathrm{mod}\,p))}(\omega\cdot(k\circ\mu))\overline{R_{(\ell^{-1}j\,(\mathrm{mod}\,p))}(\omega\cdot(\ell\circ\mu))} \\ &\times\exp\left(i\omega\cdot\frac{(k^{-1}j\,(\mathrm{mod}\,p))(k\circ\mu)-(\ell^{-1}j\,(\mathrm{mod}\,p))(\ell\circ\mu)}{p}\right). \\ =\; & p-1-\frac{p}{(p-1)^2}\sum_{k,\ell\in I'}(\mathcal{R}_{k,\mu}(\omega)\cdot\mathcal{R}_{\ell,\mu}(\omega)) \\ =\; & p-1-\frac{p}{(p-1)^2}\sum_{k\in I'}(\|\mathcal{R}_{k,\mu}(\omega)\|^2)-\frac{2p}{(p-1)^2}\sum_{\substack{k,\ell\in I'\\ k>_{\mathrm{lex}}\ell}}(\mathrm{Re}(\mathcal{R}_{k,\mu}(\omega)\cdot\mathcal{R}_{\ell,\mu}(\omega))), \end{aligned}$$

which equals

$$\begin{aligned} &\frac{p}{(p-1)^2}\sum_{k\in I'}\left(\frac{p-1}{p}-\|\mathcal{R}_{k,\mu}(\omega)\|^2\right) \\ &\qquad+\frac{2p}{(p-1)^2}\sum_{\substack{k,\ell\in I'\\ k>\ell}}\left(\frac{p-1}{p}-\mathrm{Re}(\mathcal{R}_{k,\mu}(\omega)\cdot\mathcal{R}_{\ell,\mu}(\omega))\right), \end{aligned}$$

where the identities $\frac{p}{(p-1)^2}(p-1)\left(\frac{p-1}{p}\right)=1$ and $\frac{2p}{(p-1)^2}\left(\frac{(p-1)(p-2)}{2}\right)\left(\frac{p-1}{p}\right)=p-2$ along with $|I'|=p-1$, $|\{(k,\ell)\in(I')^2 : k>\ell\}|=(p-1)(p-2)/2$ are used in the last line. Then defining $G_{k,\ell,\mu}(\omega)=\frac{p-1}{p}-\mathrm{Re}(\mathcal{R}_{k,\mu}(\omega)\cdot\mathcal{R}_{\ell,\mu}(\omega))$, we have

$$G_\mu(p\omega)=\frac{p}{(p-1)^2}\sum_{k\in I'}(f(R;\omega\cdot(k\circ\mu)))+\frac{2p}{(p-1)^2}\sum_{\substack{k,\ell\in I'\\ k>_{\mathrm{lex}}\ell}}(G_{k,\ell,\mu}(p\omega)).$$

Since $f(R;\cdot)$ is a nonnegative univariate polynomial, which has an sos representation by the Fejér-Riesz Lemma, the proof is complete if we are able to show that $G_{k,\ell,\mu}$ is a nonnegative bivariate trigonometric polynomial for every k, ℓ, μ, by Result 1.8(b). The nonnegativity is straightforward, since

$$\frac{p-1}{p} - \mathrm{Re}(\mathcal{R}_{k,\mu}(\omega) \cdot \mathcal{R}_{\ell,\mu}(\omega)) \geq \frac{p-1}{p} - \|\mathcal{R}_{k,\mu}(\omega)\| \|\mathcal{R}_{\ell,\mu}(\omega)\| \geq 0.$$

We see that

$$\begin{aligned}\mathcal{R}_{k,\mu}(\omega) \cdot \mathcal{R}_{\ell,\mu}(\omega) = \sum_{j \in I'} & R_{(k^{-1}j \,(\mathrm{mod}\, p))}(\omega \cdot (k \circ \mu)) \overline{R_{(\ell^{-1}j \,(\mathrm{mod}\, p))}(\omega \cdot (\ell \circ \mu))} \\ & \times \exp\left(i\omega \cdot \frac{(k^{-1}j \,(\mathrm{mod}\, p))(k \circ \mu) - (\ell^{-1}j \,(\mathrm{mod}\, p))(\ell \circ \mu)}{p} \right).\end{aligned}$$

Now take $x = k \circ \mu$ and $y = \ell \circ \mu$. If x and y are linearly dependent, then use Lemma 3.5 with $m = 1$ to find ζ using x as input. Then $x = a\zeta$, $y = b\zeta$, for some $a, b \in \mathbb{Z}$, and $G_{k,\ell,\mu}$ is a trigonometric polynomial in $\omega \cdot \zeta$, since

$$\begin{aligned}&(k^{-1}j \,(\mathrm{mod}\, p))(a\zeta) - (\ell^{-1}j \,(\mathrm{mod}\, p))(b\zeta) \equiv 0 \,(\mathrm{mod}\, p\mathbb{Z}^n) \\ \Longrightarrow\ & \frac{((k^{-1}j \,(\mathrm{mod}\, p))a - (\ell^{-1}j \,(\mathrm{mod}\, p))b}{p} \in \mathbb{Z},\end{aligned}$$

using Lemma 3.5(b). That is:

$$\begin{aligned}\mathcal{R}_{k,\mu}(\omega) \cdot \mathcal{R}_{\ell,\mu}(\omega) = \sum_{j \in I'} & R_{(k^{-1}j \,(\mathrm{mod}\, p))}(a(\omega \cdot \zeta)) \overline{R_{(\ell^{-1}j \,(\mathrm{mod}\, p))}(b(\omega \cdot \zeta))} \\ & \times \exp\left(i(\omega \cdot \zeta) \frac{(k^{-1}j \,(\mathrm{mod}\, p))a - (\ell^{-1}j \,(\mathrm{mod}\, p))b}{p} \right).\end{aligned}$$

is a trigonometric polynomial in $\omega \cdot \zeta$.

Otherwise, x and y are linearly independent, and we may use Lemma 3.5 with $m = 2$ to find ζ_1, ζ_2. Then if $x = a\zeta_1 + b\zeta_2$, $y = c\zeta_1 + d\zeta_2$,

$$(k^{-1}j \,(\mathrm{mod}\, p))x - (\ell^{-1}j \,(\mathrm{mod}\, p))y = [\zeta_1 | \zeta_2] \begin{bmatrix} (k^{-1}j \,(\mathrm{mod}\, p))a - (\ell^{-1}j \,(\mathrm{mod}\, p))c \\ (k^{-1}j \,(\mathrm{mod}\, p))b - (\ell^{-1}j \,(\mathrm{mod}\, p))d \end{bmatrix}$$

which is $\equiv 0 \,(\mathrm{mod}\, p\mathbb{Z}^n)$, so the latter vector is in $p\mathbb{Z}^2$, by Lemma 3.5(b). This means that the coefficients of $\omega\cdot\zeta_1$ and $\omega\cdot\zeta_2$ in the exponential are integers. That is, $\mathcal{R}_{k,\mu}(\omega)\cdot\mathcal{R}_{\ell,\mu}(\omega)$, which equals

$$\sum_{j\in I'} R_{(k^{-1}j\,(\mathrm{mod}\,p))}(a(\omega\cdot\zeta_1)+b(\omega\cdot\zeta_2))\overline{R_{(\ell^{-1}j\,(\mathrm{mod}\,p))}(c(\omega\cdot\zeta_1)+d(\omega\cdot\zeta_2))}$$
$$\times\exp\left(i[\omega\cdot\zeta_1|\omega\cdot\zeta_2]\begin{bmatrix}\frac{(k^{-1}j\,(\mathrm{mod}\,p))a-(\ell^{-1}j\,(\mathrm{mod}\,p))c}{p}\\ \frac{(k^{-1}j\,(\mathrm{mod}\,p))b-(\ell^{-1}j\,(\mathrm{mod}\,p))d}{p}\end{bmatrix}\right),$$

is a bivariate trigonometric polynomial in $\omega\cdot\zeta_1$, $\omega\cdot\zeta_2$.

This completes the proof.

□

Combining Theorem 3.3 with Results 1.6 and 1.11, we obtain the *prime coset sum method for constructing tight wavelet frames (PCSTF).*

Theorem 3.4 (PCSTF)**.** *Let R be PCSTF-admissible, and let τ be the output of PCS with input R in n dimensions. Let $g_j(p\cdot), 1\le j\le J$ be the sos generators for $f(\tau;\cdot)$ as guaranteed by Theorem 3.3. Then along with τ, the following highpass masks form a tight wavelet filter bank:*

$$q_{1,j}(\omega) := \tau(\omega)\overline{g_j(p\omega)},\quad 1\le j\le J,\ \textit{and}$$

$$q_{2,\nu}(\omega) := p^{-n/2}\exp(i\nu\cdot\omega)-\tau(\omega)\overline{\tau_\nu(p\omega)},\quad \nu\in\Gamma.$$

Therefore the wavelet system $\Lambda(\{\psi^{(i)}\})$ (c.f. (1.1)) is a tight frame for $L^2(\mathbb{R}^n)$.

We now specialize Theorem 1.1 to the tight wavelet frames constructed in Theorem 3.4.

Theorem 3.5 (VMs for PCSTF highpass masks)**.** *Let R be PCSTF-admissible, and let τ be the output of PCS with input R in n dimensions. Let τ have accuracy number a and flatness number b. Then for the highpass masks of Theorem 3.4, $q_{1,\nu},\nu\in\Gamma$ have at least a vanishing moments, and $q_{2,j}, 1\le j\le N$ have at least $\lceil\min\{2a,b\}/2\rceil$ vanishing moments.*

In particular, if R has accuracy number m, then the masks $q_{1,\nu},\nu\in\Gamma$ have at least m vanishing moments, and $q_{2,j}, 1\le j\le N$ have at least $\lceil m/2\rceil$ vanishing moments.

Proof. Since τ is interpolatory, $b\ge a$. Theorem 3.3 guarantees the existence of an sos representation for $f(\tau;\cdot)$, so we obtain the relations between the vanishing moments of the highpass masks of Theorem 3.4, $\{q_{1,\nu} : \nu \in \Gamma\}$, $\{q_{2,j} : 1 \leq j \leq N\}$, and a, b immediately from Theorem 1.1. By Result 1.5(ii), $b \geq a \geq m$ (since R is interpolatory), so we obtain the relations between the vanishing moments of these masks and m. $\square$

3.3.4 Examples

In this section, we give two examples in the case $n = 2$, $p = 3$, demonstrating our method and computing the vanishing moments of the constructed highpass masks. In both cases, the input lowpass mask has flatness and accuracy numbers equal to some positive, even integer, so the lowpass masks constructed from PCS are guaranteed to have the same flatness and accuracy numbers as the input by Corollary 3.2. We will see that the lower bounds proved in Theorems 1.1 and 3.5 are achieved in these examples.

Example 3.3. Let $p = 3$, and

$$R(\omega) = \frac{1}{9}(3 + 4\cos(\omega) + 2\cos(2\omega)).$$

Then R is PCSTF-admissible. Moreover, it's easy to see that this has accuracy and flatness numbers equal to 2, since $D^1 R(\omega) = -\frac{1}{9}(4\sin(\omega) + 4\sin(2\omega))$, which is equal to 0 at $\omega \in \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$, and $D^2 R(\omega) = -\frac{1}{9}(4\cos(\omega) + 8\cos(2\omega))$, which equals $-4/3$ at 0, and equals $2/3$ at $\omega \in \{\frac{2\pi}{3}, \frac{4\pi}{3}\}$.

Let $\Gamma = \{-1, 0, 1\}^2$. Then

$$\begin{aligned}\tau(\omega) = \frac{1}{9} &+ \frac{4}{27}(\cos(\omega_1) + \cos(\omega_2) + \cos(\omega_1 + \omega_2) + \cos(\omega_1 - \omega_2)) \\ &+ \frac{2}{27}(\cos(2\omega_1) + \cos(2\omega_2) + \cos(2(\omega_1 + \omega_2)) + \cos(2(\omega_1 - \omega_2))).\end{aligned}$$

Since $\tau_{-\nu}(\omega) = \overline{\tau_\nu(\omega)}$ for $\nu \in \Gamma'$, by Lemma 3.1 and Equation (3.2), we obtain

$$\begin{aligned} f(\tau;\omega) &= \frac{8}{9} - \frac{1}{6} \sum_{\substack{\nu\in\Gamma' \\ \nu>_{\text{lex}}0}} \left| \sum_{(\nu',j)\in\mathbf{M}(\nu)} R_j(3\omega\cdot\nu') \exp\left(i\omega\cdot(j\nu'-\nu)\right) \right|^2 \\ &= \frac{8}{9} - \frac{1}{6} \sum_{\substack{\nu\in\Gamma' \\ \nu>_{\text{lex}}0}} |R_1(3\omega\cdot\nu) + R_{-1}(3\omega\cdot(-\nu))|^2, \end{aligned}$$

where $I = \{-1,0,1\}$, and we use the Remark after Lemma 3.3 on the set $\mathbf{M}(\nu)$ in the last line. Since $R_1(\omega) = \frac{\sqrt{3}}{9}(2+\exp(-i\omega))$, and $R_{-1}(\omega) = \overline{R_1(\omega)}$, we have

$$f(\tau;\omega) = \frac{8}{9} - \frac{2}{81} \sum_{\substack{\nu\in\Gamma' \\ \nu>_{\text{lex}}0}} |2+\exp(-3i\omega\cdot\nu)|^2 = \frac{8}{81} \sum_{\substack{\nu\in\Gamma' \\ \nu>_{\text{lex}}0}} (1-\cos(3\omega\cdot\nu)),$$

and this yields

$$f(\tau;\omega) = \sum_{\substack{\nu\in\Gamma' \\ \nu>_{\text{lex}}0}} \left| \frac{2}{9}(1-\exp(-3i\omega\cdot\nu)) \right|^2.$$

Since $\tau_\nu(\omega) = \frac{1}{9}(2+\exp(-i\omega\cdot\nu))$, we obtain the highpass filters

$$\begin{aligned} q_{1,\mu}(\omega) &= \frac{2}{9}\tau(\omega)(1-\exp(3i\omega\cdot\mu)), && \mu \in M, \\ q_{2,0}(\omega) &= \frac{1}{3}(1-\tau(\omega)), \\ q_{2,\nu}(\omega) &= \frac{1}{3}\exp(i\omega\cdot\nu) - \frac{1}{9}\tau(\omega)(2+\exp(3i\omega\cdot\nu)) && \nu\in\Gamma', \end{aligned}$$

where $M = \{e_1, e_2, e_1+e_2, e_1-e_2\} = \Gamma' \cap \{k \in \mathbb{Z}^2 : k >_{\text{lex}} 0\}$.

One can easily see that the $q_{1,\mu}$ have exactly 1 vanishing moment. Clearly, $q_{2,0}$ has 2 vanishing moments (this is just the flatness number for τ). For $\nu \in \Gamma'$, we can see that $q_{2,\nu}(0) = 0$, and $D^\alpha q_{2,\nu}(0)$, $|\alpha| = 1$ is equal to $\frac{i\nu^\alpha}{3} - \frac{3i\nu^\alpha}{9} = 0$, since $\tau(0) = 1, D^\alpha\tau(0) = 0$. Thus the $q_{2,\nu}, \nu \in \Gamma'$, have at least two vanishing moments, and since $D^{(2,0)}q_{2,\nu}(0) = 2$ for $\nu \in \{e_1, e_1+e_2, e_1-e_2\}$, and $D^{(0,2)}q_{2,e_2}(0) = 2$, we see that these all have exactly two vanishing moments (using $D^{(2,0)}\tau(0) = D^{(0,2)}\tau(0) = -4/3$, $D^{(1,1)}\tau(0) = 0$). Both of these numbers match the lower bound given by Theorem 3.5.

The filter coefficient diagrams for these masks are given in Figures 3.2 and 3.3, where the boldface number indicates the origin, and the grid of numbers show the filter coefficients for the corresponding mask in the plane.

Note that the filters for $q_{2,\nu}, \nu \in \{-e_1, e_2, -e_2\}$ are just the corresponding rotation of q_{2,e_1} shown in Figure 3.2(c), and the filters for $q_{2,\nu}, \nu \in \{-(e_1+e_2), e_1-e_2, -(e_1-e_2)\}$ are just the corresponding rotation of q_{2,e_1+e_2} shown in Figure 3.2(d), so we do not show these additional filters. The same reasoning is used for Figure 3.3 as well. ■

1/3	0	1/3	0	1/3
0	2/3	2/3	2/3	0
1/3	2/3	**1**	2/3	1/3
0	2/3	2/3	2/3	0
1/3	0	1/3	0	1/3

(a) $\tau(\omega)$

-1/9	0	-1/9	0	-1/9
0	-2/9	-2/9	-2/9	0
-1/9	-2/9	**8/3**	-2/9	-1/9
0	-2/9	-2/9	-2/9	0
-1/9	0	-1/9	0	-1/9

(b) $q_{2,(0,0)}(\omega)$

-1/27	0	-1/27	-2/27	-1/27	-2/27	0	-2/27
0	-2/27	-2/27	-2/27	-4/27	-4/27	-4/27	0
-1/27	-2/27	-1/9	-4/27	76/27	**-2/9**	-4/27	-2/27
0	-2/27	-2/27	-2/27	-4/27	-4/27	-4/27	0
-1/27	0	-1/27	-2/27	-1/27	-2/27	0	-2/27

(c) $q_{2,(1,0)}(\omega)$

			-2/27	0	-2/27	0	-2/27
			0	-4/27	-4/27	-4/27	0
			-2/27	-4/27	**-2/9**	-4/27	-2/27
-1/27	0	-1/27	0	76/27	-4/27	-4/27	0
0	-2/27	-2/27	-4/27	0	-2/27	0	-2/27
-1/27	-2/27	-1/9	-2/27	-1/27			
0	-2/27	-2/27	-2/27	0			
-1/27	0	-1/27	0	-1/27			

(d) $q_{2,(1,1)}(\omega)$

Figure 3.2: Wavelet and lowpass filters from Example 3.3

Example 3.4. Let

$$R(\omega) = \frac{1}{243}(81 + 120\cos(\omega) + 60\cos(2\omega) - 10\cos(4\omega) - 8\cos(5\omega)),$$

which is a lowpass mask with prime dilation 3. A calculation reveals that R has accuracy and flatness numbers both equal to 4, and R is clearly interpolatory and PCSTF-admissible. Then letting τ be the output of PCS with input R for any choice of n and Γ, Corollary 3.2 tells us that the accuracy and flatness numbers of τ are also both equal to 4.

-2/27	0	-2/27	2/27	-2/27	2/27	0	2/27
0	-4/27	-4/27	-4/27	4/27	4/27	4/27	0
-2/27	-4/27	-2/9	-2/27	2/27	**2/9**	4/27	2/27
0	-4/27	-4/27	-4/27	4/27	4/27	4/27	0
-2/27	0	-2/27	2/27	-2/27	2/27	0	2/27

(a) $q_{1,(1,0)}$

			2/27	0	2/27	0	2/27
			0	4/27	4/27	4/27	0
			2/27	4/27	**2/9**	4/27	2/27
-2/27	0	-2/27	0	2/27	4/27	4/27	0
0	-4/27	-4/27	-2/27	0	2/27	0	2/27
-2/27	-4/27	-2/9	-4/27	-2/27			
0	-4/27	-4/27	-4/27	0			
-2/27	0	-2/27	0	-2/27			

(b) $q_{1,(1,1)}(\omega)$

Figure 3.3: Wavelet filters from Example 3.3

Choosing $n = 2$ and $\Gamma = \{-1, 0, 1\}^2$, we see that

$$\tau(\omega) = \frac{1}{9} + \frac{1}{729} \sum_{\substack{\nu \in \Gamma' \\ \nu >_{\text{lex}} 0}} (120\cos(\omega \cdot \nu) + 60\cos(2\omega \cdot \nu) - 10\cos(4\omega \cdot \nu) - 8\cos(5\omega \cdot \nu)).$$

This gives $\tau_\nu(\omega) = \frac{1}{243}(-5\exp(i\omega \cdot \nu) + 60 + 30\exp(-i\omega \cdot \nu) - 4\exp(-2i\omega \cdot \nu))$ for all $\nu \in \Gamma'$, and

$$\begin{aligned} f(\tau; \omega) &= \frac{40}{3^{10}} \sum_{\substack{\nu \in \Gamma' \\ \nu >_{\text{lex}} 0}} (101 - 138\cos(3\omega \cdot \nu) + 39\cos(6\omega \cdot \nu) - 2\cos(9\omega \cdot \nu)) \\ &=: \frac{1}{9} \sum_{\nu \in M} G_\nu(3\omega), \end{aligned}$$

where $M = \{e_1, e_2, e_1 + e_2, e_1 - e_2\}$. Letting $\tilde{G}$ be the univariate polynomial such that $G_\nu(\omega) = \tilde{G}(\omega \cdot \nu)$, we see that $\tilde{G}(\omega) = \frac{20}{3^8}(2(1-\cos(\omega)))^2(31-4\cos(\omega))$, for $\omega \in \mathbb{T}$. Moreover, searching for $\alpha, \beta \in \mathbb{C}$ such that $|\alpha + \beta \exp(i\omega)|^2 = 31 - 4\cos(\omega)$ yields $\alpha = (\sqrt{27} + \sqrt{35})/2$ and $\beta = (\sqrt{27} - \sqrt{35})/2$. Then, since $2(1-\cos(\omega)) = |1-\exp(i\omega)|^2$, with $a = 5\sqrt{7}$, $b = \sqrt{15}$,

$\tilde{G}(\omega)$ equals

$$\begin{aligned}&\frac{5}{3^8}\left||1-\exp(i\omega)|^2(\sqrt{35}+\sqrt{27}-(\sqrt{35}-\sqrt{27})\exp(i\omega))\right|^2\\&=\left|\frac{1}{81}\left((a+3b)\exp(-i\omega)-3(a+b)+3(a-b)\exp(i\omega)-(a-3b)\exp(2i\omega)\right)\right|^2.\end{aligned}$$

The highpass masks satisfying the UEP conditions with τ are given by

$$\begin{aligned}q_{1,\mu}(\omega)&=\frac{\tau(\omega)}{243}((a+3b)\exp(3i\omega\cdot\mu)-3(a+b)+3(a-b)\exp(-3i\omega\cdot\mu))\\&\quad-\frac{\tau(\omega)}{243}(a-3b)\exp(-6i\omega\cdot\mu),\quad \mu\in\{e_1,e_2,e_1+e_2,e_1-e_2\}=M,\\q_{2,\nu}(\omega)&=\frac{1}{3}\exp(i\omega\cdot\nu)-\tau(\omega)\overline{\tau_\nu(3\omega)},\quad \nu\in\Gamma.\end{aligned}$$

We can clearly see that all of the $q_{1,\mu}$ have exactly 2 vanishing moments by our computation of $\tilde{G}$ above. The $q_{2,\nu}$ all have at least 4 vanishing moments, and $q_{2,0}$ has exactly 4 because this is just the flatness number of τ. For $\nu\in\Gamma'$, using the calculation in the proof of Proposition 1.1, when $|\alpha|=4$,

$$D^\alpha q_{2,\nu}(0)=\frac{\nu^\alpha}{3}-\overline{D^\alpha[\tau_\nu(3\omega)]_{\omega=0}}-\frac{1}{3}D^\alpha\tau(0),$$

using $D^\beta\tau(0)=\delta(\beta)$ for $|\beta|\leq 3$. Since

$$D^\alpha\tau_\nu(3\omega)|_{\omega=0}=\frac{(3i)^{|\alpha|}\nu^\alpha}{243}(-5+30(-1)^{|\alpha|}-4(-2)^{|\alpha|}),$$

which equals $\nu^\alpha(-13)$, and $D^{(4,0)}\tau(0)=D^{(0,4)}\tau(0)=-80/3$, we see that for $\alpha\in\{(4,0),(0,4)\}$, $D^\alpha q_{2,\nu}(0)=\nu^\alpha(40/3)+80/9$, which can be made nonzero for some choice of α in this set for each $\nu\in\Gamma'$. Thus the $q_{2,\nu}$ have exactly 4 vanishing moments for $\nu\in\Gamma$. ■

3.4 Summary

In this chapter, we developed the prime coset sum method for constructing tight wavelet frames, a novel method for generating nonseparable tight wavelet frames with prime dilation, using the theory of sos representations for nonnegative trigonometric polynomials. We studied the vanishing moments of the wavelets resulting from our method, and we proved new results about the accuracy and flatness numbers of lowpass masks arising from the prime coset sum method.

The idea of orbit decompositions and the lemmas from lattice theory were used in our setting to decompose $f(\tau;\cdot)$ into nonnegative components that could be written as a univariate or bivariate trigonometric polynomial in some appropriate variable or variables. These ideas can be extended to more general dilation matrices than those considered here, and this may be a fruitful approach for finding sos representations in those cases. This is most likely to be successful in cases where there is some symmetry to exploit related to this structure, as there is in the case of PCS-generated lowpass masks.

Chapter 4

Sums of Squares Representations

In this chapter, we take a brief detour from wavelet construction to prove the existence of sums of rational squares representations for nonnegative trigonometric polynomials in any number of variables. This has been shown in two dimensions [5], where in fact sos representations are known to exist, as in Result 1.8(b), but was not known in the general case. It has also been shown that for a positive trigonometric polynomial in any number of variables, there is a sum of squares representation, which is Result 1.8(d). The inspiration for this result was to reconcile the theory in the case of trigonometric polynomials with that of ordinary polynomials with real coefficients, where it is known that sums of rational squares representations exist for any number of variables, as in Artin's Result 1.9(a). This inspiration ends up going much further than motivation, however, since we end up applying Artin's theorem in our proof, a significant portion of which may be thought of as showing that the unit circle in $\mathbb{C}$ is not so different from $\mathbb{R}$. In particular, we find a rational polynomial which takes the unit circle minus a point to the real line, which is invertible and has so many symmetries that it takes nonnegative trigonometric polynomials to nonnegative rational polynomials on the real line with real coefficients.

In Section 4.2, we then extend this result to positive semidefinite matrices with trigonometric polynomial entries, since the techniques required are similar. The major result we want to apply in this case is Result 1.10, and in fact, we are free to do this when the matrix has real-valued entries, but in order to obtain the theorem in its full generality, we need to extend this result to the case of matrices with potentially complex-valued entries. This will actually go through quite smoothly, and the differences between these two cases resembles in many ways the extension from real symmetric to Hermitian matrices.

The results in this chapter on sums of rational squares (sors) representations for nonnegative trigonometric polynomials will be applied in Chapter 5 to establish equivalent conditions for the existence of rational highpass masks satisfying the oblique extension principle conditions with a certain lowpass mask and vanishing moment recovery function. While there are a few places where the matrix version of this result could be applied, we will frequently require additional structure for the matrices we are constructing in that section, so we will construct the needed representations more explicitly.

4.1 Sors Representations for Nonnegative Rational Polynomials on $(\partial\mathbb{D})^n$

In this section, our goal is to prove one of our main theorems, which says that a nonnegative trigonometric polynomial in any number of variables has an sors representation.

The proof of this theorem will proceed in three major steps. First, we will define a particular Möbius transformation $\mu \in \mathbb{C}(z_1)$ which maps $\partial\mathbb{D}$ to $\mathbb{R}$, and which has an inverse also in $\mathbb{C}(z_1)$. We use this map μ to define the induced map $M : \mathbb{C}(\mathbf{z}) \to \mathbb{C}(\mathbf{z})$ by $M(f)(z) = f(\mu(z_1), \dots, \mu(z_n))$, along with its inverse, and we then prove several properties of μ and the induced map, as well as their inverses. Finally, we will show that for $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ satisfying the assumptions of the theorem, $M(f)$ belongs to $\mathbb{R}(\mathbf{z})$ and is nonnegative on $\mathbb{R}^n$, so that it has an sors representation of functions in $\mathbb{R}(\mathbf{z})$ by Result 1.9(a). We will then apply the inverse of M to each of these generators, and show that this leads to an sors representation of f on $(\partial\mathbb{D})^n$.

Theorem 4.1. *Let $f \in \mathbb{C}(\mathbf{z})$ be such that $f(z) \geq 0$ for all $z \in (\partial\mathbb{D})^n$ at which it is defined. Then f is an sors on $(\partial\mathbb{D})^n$ of at most 2^n functions in $\mathbb{C}(\mathbf{z})$, i.e., there exist $g_j \in \mathbb{C}(\mathbf{z})$, $1 \leq j \leq J \leq 2^n$ such that*

$$f(z) = \sum_{j=1}^{J} |g_j(z)|^2, \quad \textit{for all } z \in (\partial\mathbb{D})^n \textit{ where } f(z) \textit{ is defined.}$$

When we write the expanded form of a polynomial or Laurent polynomial $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$, we will frequently use the shorthand notations $f(z) = \sum_\alpha f_\alpha z^\alpha$, $\sum_{\alpha >_{\text{lex}} 0} f_\alpha z^\alpha$, or $\sum_{\alpha \geq_{\text{e}} 0} f_\alpha z^\alpha$, which in each case refers to a sum over an appropriate finite subset of $\mathbb{Z}^n$ (see Section 1.3.4 for more details on the orderings $\geq_{\text{lex}}$, $\geq_{\text{e}}$).

We introduce a notation for the rational polynomial obtained by inverting each variable below, since this is an operation we will consider several times in what follows.

Definition 4.1. Given $f \in \mathbb{C}(\mathbf{z})$, we define $\tilde{f}(z_1, \ldots, z_n) := f(z_1^{-1}, \ldots, z_n^{-1})$, and denote by $\mathbb{R}(\mathbf{z})_\sim$ the subfield of $\mathbb{R}(\mathbf{z})$ of all elements f with $f = \tilde{f}$. $\square$

For $f \in \mathbb{C}(\mathbf{z})$, it is clear that $\tilde{f}(z) = f(\bar{z})$ for all $z \in (\partial\mathbb{D})^n$. When the coefficients are real, this gives $\tilde{f}(z) = \overline{f(z)}$ for all $z \in (\partial\mathbb{D})^n$. Thus, for $f \in \mathbb{R}(\mathbf{z})$, $f \in \mathbb{R}(\mathbf{z})_\sim$ if and only if f is real-valued on $(\partial\mathbb{D})^n$.

Lemma 4.1 (Alternative characterization for $\mathbb{R}(\mathbf{z})_\sim$)**.** *If $f \in \mathbb{R}(\mathbf{z})$, $f = p/q$, where $p, q \in \mathbb{R}[\mathbf{z}]$, $q \neq 0$, then $f \in \mathbb{R}(\mathbf{z})_\sim$ if and only if $p\tilde{q} \in \mathbb{R}(\mathbf{z})_\sim$. If $f \in \mathbb{R}[\mathbf{z}^{\pm 1}]$, $f = \sum_{\alpha \in \mathbb{Z}^n} f_\alpha z^\alpha$, then $f \in \mathbb{R}(\mathbf{z})_\sim$ if and only if $f_\alpha = f_{-\alpha}$ for all $\alpha \in \mathbb{Z}^n$.*

Proof. In the first case, $f = (p\tilde{q})/(q\tilde{q})$, and since $q\tilde{q} \in \mathbb{R}(\mathbf{z})_\sim$, $f = \tilde{f}$ if and only if $p\tilde{q} = \widetilde{(p\tilde{q})}$. In the second case, $\tilde{f} = \sum_\alpha f_\alpha z^{-\alpha} = \sum_\alpha f_{-\alpha} z^\alpha$, so $f - \tilde{f} = \sum_\alpha (f_\alpha - f_{-\alpha}) z^\alpha$ equals 0 if and only if $f_\alpha = f_{-\alpha}$ for all $\alpha \in \mathbb{Z}^n$. $\square$

Lemma 4.2 (Nonvanishing on sufficiently large rectangles)**.** *Let $S_1, \ldots, S_n \subseteq \mathbb{C}$ such that for each $1 \leq j \leq n$, the cardinality of S_j is infinite. If $f \in \mathbb{C}[\mathbf{z}]$ is such that $f(z) = 0$ for all $z \in S_1 \times \cdots \times S_n$, $f = 0$. If $f \in \mathbb{C}(\mathbf{z})$ is such that $f(z)$ is defined and equal to 0 for all $z \in S_1 \times \cdots \times S_n$, $f = 0$.*

Proof. The first case is just [43, Ch. IV.1, Corollary 1.6]. If $f = p/q \in \mathbb{C}(\mathbf{z})$ is in lowest terms, then the assumptions on f imply that $p(z) = 0$ for all $z \in S_1 \times \cdots \times S_n$, so $p = 0$, and thus $f = 0$. $\square$

Definition 4.2. We define the *forward Möbius transformation* $\mu \in \mathbb{C}(z_1)$ by

$$\mu(z_1) = \frac{z_1 + i}{i(z_1 - i)} = \frac{2 - i(z_1 - z_1^{-1})}{z_1 + z_1^{-1}}. \tag{4.1}$$

We call $\tilde{\mu}$ the *reverse Möbius transformation*, since it is the inverse of μ (see Lemma 4.3(a) immediately below). For $f \in \mathbb{C}(\mathbf{z})$, let $M(f)(z) = f(\mu(z_1), \ldots, \mu(z_n))$, which we call the *extended forward Möbius transformation*, and $\widetilde{M}(f)(z) = f(\tilde{\mu}(z_1), \ldots, \tilde{\mu}(z_n))$ the *extended reverse Möbius transformation*. $\square$

The latter equality in (4.1) follows from multiplying and dividing by $(-i)(1 + iz_1^{-1})$, since

$$\frac{z_1 + i}{i(z_1 - i)} \frac{(-i)(1 + iz_1^{-1})}{(-i)(1 + iz_1^{-1})} = \frac{(-i)(z_1 + i + i - z_1^{-1})}{z_1 - i + i + z_1^{-1}} = \frac{2 - i(z_1 - z_1^{-1})}{z_1 + z_1^{-1}}.$$

We note that similar maps are used in [21] and elsewhere, but the map μ that we have chosen has several additional symmetries which are important for obtaining the result. Since we will be switching between the domains $\mathbb{C}, \partial\mathbb{D}$, and $\mathbb{R}$, we will use z when we think about evaluating on some subregion of $\mathbb{C}$, and x when we think about evaluating on $\mathbb{R}$.

Lemma 4.3 (Properties of $\mu, \tilde{\mu}$)**.** *Let μ be the forward Möbius transformation.*

(a) μ maps $\mathbb{C} \setminus \{i\}$ continuously and bijectively to $\mathbb{C} \setminus \{-i\}$ with continuous inverse $\tilde{\mu}$ on this set. μ has a simple pole at i, and $\tilde{\mu}$ has a simple pole at $-i$.

(b) $\mu\tilde{\mu} = 1$.

(c) For all $z \in \mathbb{C} \setminus \{i\}$, $\overline{\mu(z)} = \tilde{\mu}(\bar{z})$.

(d) For $x \in \mathbb{R}$, $\mu(x), \tilde{\mu}(x) \in \partial\mathbb{D}$.

(e) For $z \in \partial\mathbb{D} \setminus \{i\}$, $\mu(z) \in \mathbb{R}$. For $z \in \partial\mathbb{D} \setminus \{-i\}$, $\tilde{\mu}(z) \in \mathbb{R}$.

Proof. For $z_1 \in \mathbb{C}$, $\tilde{\mu}(z_1) = \frac{z_1^{-1}+i}{i(z_1^{-1}-i)} = \frac{1+iz_1}{i(1-iz_1)} = \frac{z_1-i}{(-i)(z_1+i)} = \frac{2+i(z_1-z_1^{-1})}{z_1+z_1^{-1}}$, where the last equality follows from multiplying and dividing by $i(1 - iz_1^{-1})$.

(a) We have

$$\begin{aligned} \mu(\tilde{\mu}(z_1)) &= \frac{\tilde{\mu}(z_1) + i}{i(\tilde{\mu}(z_1) - i)} \\ &= \frac{z_1 - i + i(-i)(z_1 + i)}{i(z_1 - i - i(-i)(z_1 + i))} \\ &= \frac{2z_1}{2} = z_1, \end{aligned} \qquad \begin{aligned} \tilde{\mu}(\mu(z_1)) &= \frac{\mu(z_1) - i}{(-i)(\mu(z_1) + i)} \\ &= \frac{z_1 + i - i(i)(z_1 - i)}{(-i)(z_1 + i + i(i)(z_1 - i))} \\ &= \frac{2z_1}{2} = z_1, \end{aligned}$$

where the first formula is valid for evaluation whenever $z \in \mathbb{C} \setminus \{-i\}$ and the second is valid whenever $z \in \mathbb{C} \setminus \{i\}$, but it is clear that the singularity is removable in either case. This proves that the image of μ is $\mathbb{C} \setminus \{-i\}$, and the image of $\tilde{\mu}$ is $\mathbb{C} \setminus \{i\}$, so that μ maps $\mathbb{C} \setminus \{i\}$ bijectively to $\mathbb{C} \setminus \{-i\}$, the set on which $\tilde{\mu}$ is defined, and $\tilde{\mu}$ gives the inverse.

(b) We have $\mu(z_1)\tilde{\mu}(z_1) = \frac{(z_1+i)(z_1-i)}{i(z_1-i)(-i)(z_1+i)} = 1$ (with removable singularities at $\pm i$).

(c) From the definition and computation of $\tilde{\mu}$ above, we see that $\tilde{\mu}(\bar{z}) = \overline{\mu(z)}$ for all $z \in \mathbb{C} \setminus \{i\}$.

(d) Note that whenever $x \in \mathbb{R}$, $\overline{\mu(x)} = \overline{\mu(\bar{x})} = \tilde{\mu}(x)$, so applying (b), $1 = \mu(x)\tilde{\mu}(x) = |\mu(x)|^2$, which proves that $\mu(x) \in \partial\mathbb{D}$. Since for $x \in \mathbb{R}$, $\tilde{\mu}(x) = \overline{\mu(x)}$, $\tilde{\mu}(x)$ is also in $\partial\mathbb{D}$.

(e) Whenever $z \in (\partial\mathbb{D} \setminus \{i\})$, $\overline{\mu(z)} = \tilde{\mu}(\bar{z}) = \tilde{\mu}(z^{-1}) = \mu(z)$, so $\mu(z) \in \mathbb{R}$. Similarly, for $z \in (\partial\mathbb{D} \setminus \{-i\})$, $\tilde{\mu}(z) \in \mathbb{R}$.

$\square$

Now we extend $\mu, \tilde{\mu}$ to $\mathbb{C}(\mathbf{z})$.

Lemma 4.4 (Properties of $M, \widetilde{M}$)**.** *Let $f \in \mathbb{C}(\mathbf{z})$, and let $M, \widetilde{M}$ be the extended forward and reverse Möbius transformations, respectively.*

(a) $M(f), \widetilde{M}(f) \in \mathbb{C}(\mathbf{z})$;

(b) $M(\widetilde{M}(f)) = \widetilde{M}(M(f)) = f$;

(c) $\{z \in (\mathbb{C} \setminus \{i\})^n : M(f)(z) = 0\} = \{(\tilde{\mu}(z_1), \ldots, \tilde{\mu}(z_n)) : z \in (\mathbb{C} \setminus \{-i\})^n, f(z) = 0\}$;

(d) $\{z \in \mathbb{C}^n : M(f)(z) \textit{ is undefined}\} \subseteq \{z \in \mathbb{C}^n : z_j = i \textit{ for some } 1 \leq j \leq n\} \cup \{(\tilde{\mu}(z_1), \ldots, \tilde{\mu}(z_n)) : z \in (\mathbb{C} \setminus \{-i\})^n, f(z) \textit{ is undefined}\}$;

(e) $M(\tilde{f}) = \widetilde{M}(f) = \widetilde{M(f)}$;

(f) If $f \in \mathbb{R}(\mathbf{z})$, *for* $z \in \mathbb{C}^n$, $\overline{M(f)(z)} = \widetilde{M}(f)(\bar{z})$ *whenever* $M(f)(z)$ *is defined;*

(g) If $f \in \mathbb{R}(\mathbf{z})$, *for* $x \in \mathbb{R}^n$, $\overline{M(f)(x)} = \widetilde{M}(f)(x)$ *whenever* $M(f)(x)$ *is defined;*

(h) If $f \in \mathbb{R}(\mathbf{z})$, for $z \in (\partial\mathbb{D})^n$, $\overline{M(f)(z)} = M(f)(z)$, and $\overline{\widetilde{M}(f)(z)} = \widetilde{M}(f)(z)$, whenever $M(f)(z)$ and $\widetilde{M}(f)(z)$ are respectively defined;

(i) If $f \in \mathbb{R}(\mathbf{z})_\sim$, then $M(f) = \widetilde{M}(f) \in \mathbb{R}(\mathbf{z})_\sim$;

(j) If $f \in \mathbb{R}(\mathbf{z})$, $(\pm i)M(f - \tilde{f}) \in \mathbb{R}(\mathbf{z})$; and

(k) If $f(z) \geq 0$ for all $z \in (\partial\mathbb{D})^n$ at which it is defined, then $M(f)(x) \geq 0$ for all $x \in \mathbb{R}^n$ at which it is defined.

Proof. (a) If $f \in \mathbb{C}[\mathbf{z}]$, $f = \sum_{\alpha \geq_e 0} f_\alpha z^\alpha$, $f_\alpha \in \mathbb{C}$ for all $\alpha \in \mathbb{Z}^n$, then from Equation (4.1),

$$M(f) = \sum_{\alpha \geq_e 0} f_\alpha \prod_{j=1}^{n} \left(\frac{z_j + i}{i(z_j - i)} \right)^{\alpha_j}$$

clearly belongs to $\mathbb{C}(\mathbf{z})$. If $M(f) = 0$, then $0 = M(f)(x) = f(\mu(x_1), \ldots, \mu(x_n))$ for all $x \in \mathbb{R}^n$, which means that $f(z) = 0$ for all $z \in (\partial\mathbb{D} \setminus \{-i\})^n$ by Lemma 4.3(a) and (d). By Lemma 4.2, with $S_j = \partial\mathbb{D} \setminus \{-i\}$ for all j, this means that $f = 0$. Then if $f = p/q \in \mathbb{C}(\mathbf{z})$ is in lowest terms, $q \neq 0$, so $M(f) = M(p)/M(q) \in \mathbb{C}(\mathbf{z})$, since we have just seen that $M(q) \neq 0$. The argument for $\widetilde{M}$ is similar.

(b) $M(\widetilde{M}(f)) = M(f(\tilde{\mu}(z_1), \ldots, \tilde{\mu}(z_n))) = f(\tilde{\mu}(\mu(z_1)), \ldots, \tilde{\mu}(\mu(z_n))) = f(z_1, \ldots, z_n) = f$, by the calculation in the proof of Lemma 4.3(a). The case of $\widetilde{M}(M(f)) = f$ is similar, using the other calculation at the beginning of that proof.

(c) By Lemma 4.3(a), $(\mathbb{C} \setminus \{i\})^n = (\tilde{\mu}(\mathbb{C} \setminus \{-i\}))^n$. By (b), for $z' \in (\mathbb{C} \setminus \{-i\})^n$, $f(z') = \widetilde{M}(M(f))(z') = M(f)(\tilde{\mu}(z'_1), \ldots, \tilde{\mu}(z'_n))$. Then given a point $z \in (\mathbb{C} \setminus \{i\})^n$ with $M(f)(z) = 0$, $z = (\tilde{\mu}(z'_1), \ldots, \tilde{\mu}(z'_n))$ for some $z' \in (\mathbb{C} \setminus \{-i\})^n$, and $f(z') = 0$; conversely, for $z' \in (\mathbb{C} \setminus \{-i\})^n$ with $f(z') = 0$, $z = (\tilde{\mu}(z'_1), \ldots, \tilde{\mu}(z'_n))$ satisfies $M(f)(z) = 0$.

(d) Since μ is not defined at i, $M(f)(z)$ may not be defined if $z_j = i$ for some $1 \leq j \leq n$. Otherwise, letting $f = p/q$ be in lowest terms, we can apply (c) to the set where $M(q)(z) = 0$, since $M(f) = M(p)/M(q)$.

(e) $M(\tilde{f}) = \tilde{f}(\mu(z_1), \ldots, \mu(z_n)) = f((\mu(z_1))^{-1}, \ldots, (\mu(z_n))^{-1}) = f(\tilde{\mu}(z_1), \ldots, \tilde{\mu}(z_n)) = \widetilde{M}(f)$, by Lemma 4.3(b). $\widetilde{M(f)} = M(f)(z_1^{-1}, \ldots, z_n^{-1}) = f(\mu(z_1^{-1}), \ldots, \mu(z_n^{-1})) =$

$f(\tilde{\mu}(z_1), \dots, \tilde{\mu}(z_n)) = \widetilde{M}(f)$.

(f) Let $f \in \mathbb{R}[\mathbf{z}]$, $f = \sum_{\alpha \geq_e 0} f_\alpha z^\alpha$, $f_\alpha \in \mathbb{R}$ for all $\alpha \in \mathbb{Z}^n$. Then by Equation (4.1),

$$M(f) = \sum_{\alpha \geq_e 0} f_\alpha \prod_{j=1}^{n} \left(\frac{z_j + i}{i(z_j - i)} \right)^{\alpha_j},$$

so

$$\overline{M(f)(z)} = \sum_{\alpha \geq_e 0} f_\alpha \prod_{j=1}^{n} \left(\frac{\bar{z}_j - i}{(-i)(\bar{z}_j + i)} \right)^{\alpha_j} = \widetilde{M}(f)(\bar{z}),$$

using the computation of $\tilde{\mu}$ at the beginning of the proof of Lemma 4.3. Now if $f = p/q$ is in lowest terms, $\overline{M(f)(z)} = \overline{M(p)(z)}/\overline{M(q)(z)} = \widetilde{M}(p)(\bar{z})/\widetilde{M}(q)(\bar{z}) = \widetilde{M}(f)(\bar{z})$.

(g) From (f), $\overline{M(f)(x)} = \widetilde{M}(f)(\bar{x}) = \widetilde{M}(f)(x)$, where the last equality follows since $x \in \mathbb{R}^n$.

(h) From (f), $\overline{M(f)(z)} = \widetilde{M}(f)(\bar{z}) = \widetilde{M}(f)(z_1^{-1}, \dots, z_n^{-1})$, since $z \in (\partial\mathbb{D})^n$. This equals $\widetilde{\widetilde{M}(f)}(z) = M(f)(z)$ by (e), and the proof is similar for the case with $\widetilde{M}$.

(i) By (e), since $f \in \mathbb{R}(\mathbf{z})_\sim$, $M(f) = M(\tilde{f}) = \widetilde{M(f)}$, so $M(f) \in \mathbb{R}(\mathbf{z})_\sim$ if it is in $\mathbb{R}(\mathbf{z})$, which we now verify. From Lemma 4.1, if $f = p/q \in \mathbb{R}(\mathbf{z})_\sim$, then $p\tilde{q} \in \mathbb{R}(\mathbf{z})_\sim$, so if we can show that $M(p\tilde{q}), M(q\tilde{q})$ are in $\mathbb{R}(\mathbf{z})_\sim$, $M(f) = M(p\tilde{q})/M(q\tilde{q}) \in \mathbb{R}(\mathbf{z})_\sim$ also. Thus, it suffices to consider the case where $f \in \mathbb{R}[\mathbf{z}^{\pm 1}]$. Again from Lemma 4.1, this means that $f_\alpha = f_{-\alpha}$ for all $\alpha \in \mathbb{Z}^n$, so $f = f_0 + \sum_{\alpha >_{\text{lex}} 0} f_\alpha (z^\alpha + z^{-\alpha})$, and $M(f) = f_0 + \sum_{\alpha >_{\text{lex}} 0} f_\alpha (((\mu(z_j))_j)^\alpha + ((\mu(z_j))_j)^{-\alpha})$. Thus, it suffices to show that $((\mu(z_j))_j)^\alpha + ((\mu(z_j))_j)^{-\alpha} \in \mathbb{R}(\mathbf{z})$ whenever $\alpha >_{\text{lex}} 0$.

For $\alpha >_{\text{lex}} 0$, let $\varepsilon \in \{-1, 1\}^n$ be such that $\epsilon_j = 1$ if $\alpha_j \geq 0$ and is -1 otherwise, and let $\beta = (|\alpha_j|)_j$. By Lemma 4.3(b), we have $\mu(z_j)^{\alpha_j} = \tilde{\mu}(z_j)^{-\alpha_j}$, so using the definition of μ (4.1) and the calculation of $\tilde{\mu}$ at the beginning of the proof of Lemma 4.3, we see that

$$\mu(z_j)^{\alpha_j} = \left(\frac{2 - \epsilon_j i(z_j - z_j^{-1})}{z_j + z_j^{-1}} \right)^{\beta_j}.$$

Using this, we compute

$$\begin{aligned}
&((\mu(z_j))_j)^{\alpha} + ((\mu(z_j))_j)^{-\alpha} \\
&= \prod_{j=1}^{n} \left(\frac{2 - \varepsilon_j i(z_j - z_j^{-1})}{z_j + z_j^{-1}} \right)^{\beta_j} + \prod_{j=1}^{n} \left(\frac{2 + \varepsilon_j i(z_j - z_j^{-1})}{z_j + z_j^{-1}} \right)^{\beta_j} \\
&= ((z_j + z_j^{-1})_j)^{-\beta} \left(\prod_{j=1}^{n} (2 - \varepsilon_j i(z_j - z_j^{-1}))^{\beta_j} + \prod_{j=1}^{n} (2 + \varepsilon_j i(z_j - z_j^{-1}))^{\beta_j} \right).
\end{aligned}$$

Now we apply Equation (1.11) to both of these products, obtaining for the right factor of the above expression

$$\sum_{0 \leq_{\mathrm{e}} k \leq_{\mathrm{e}} \beta} \binom{\beta}{k} 2^{|\beta|-|k|} ((\varepsilon_j (z_j - z_j^{-1}))_j)^k ((-i)^{|k|} + i^{|k|}).$$

Since $(-i)^{|k|} + i^{|k|}$ is always in $\mathbb{R}$, we see that this is in $\mathbb{R}(\mathbf{z})$, which completes the proof, since the factor $((z_j + z_j^{-1})_j)^{-\beta} \in \mathbb{R}(\mathbf{z})$.

(j) For $f = p/q \in \mathbb{R}(\mathbf{z})$, $f - \tilde{f} = (p\tilde{q} - \tilde{p}q)/(q\tilde{q})$, so $M(f - \tilde{f}) = M(p\tilde{q} - \tilde{p}q)/M(q\tilde{q})$, and since $q\tilde{q} \in \mathbb{R}(\mathbf{z})_{\sim}$, by (i), $M(q\tilde{q}) \in \mathbb{R}(\mathbf{z})_{\sim}$, so it suffices to show that $(\pm i)M(f - \tilde{f}) \in \mathbb{R}(\mathbf{z})$ when $f \in \mathbb{R}[\mathbf{z}^{\pm 1}]$.

Letting $f = \sum_{\alpha} f_{\alpha} z^{\alpha}$, we see that $f - \tilde{f} = \sum_{\alpha} (f_{\alpha} - f_{-\alpha}) z^{\alpha} = \sum_{\alpha >_{\mathrm{lex}} 0} (f_{\alpha} - f_{-\alpha})(z^{\alpha} - z^{-\alpha}) = \sum_{\alpha >_{\mathrm{lex}} 0} c_{\alpha}(z^{\alpha} - z^{-\alpha})$, where $c_{\alpha} \in \mathbb{R}$ for all $\alpha \in \mathbb{Z}^n$, $\alpha >_{\mathrm{lex}} 0$. This gives $M(f - \tilde{f}) = \sum_{\alpha >_{\mathrm{lex}} 0} c_{\alpha}(((\mu(z_j))_j)^{\alpha} - ((\mu(z_j))_j)^{-\alpha})$, so we need to show that for all $\alpha \in \mathbb{Z}^n$, $\alpha >_{\mathrm{lex}} 0$, $(\pm i)(((\mu(z_j))_j)^{\alpha} - ((\mu(z_j))_j)^{-\alpha}) \in \mathbb{R}(\mathbf{z})$.

For $\alpha >_{\mathrm{lex}} 0$, we proceed as in the proof of (i) to see that $((\mu(z_j))_j)^{\alpha} - ((\mu(z_j))_j)^{-\alpha}$ equals

$$((z_j + z_j^{-1})_j)^{-\beta} \sum_{0 \leq_{\mathrm{e}} k \leq_{\mathrm{e}} \beta} \binom{\beta}{k} 2^{|\beta|-|k|} ((\varepsilon_j (z_j - z_j^{-1}))_j)^k ((-i)^{|k|} - i^{|k|}).$$

Since $(-i)^{|k|} - i^{|k|} = 0$ whenever $|k|$ is even, the right factor equals

$$i\left(\sum_{\substack{0\leq_{\mathrm{e}} k\leq_{\mathrm{e}}\beta \\ |k| \text{ odd}}} \binom{\beta}{k} 2^{|\beta|-|k|+1}((\varepsilon_j(z_j - z_j^{-1}))_j)^k(-1)^{(|k|+1)/2}\right),$$

which is clearly in $\mathbb{R}(\mathbf{z})$ when multiplied by $\pm i$. Since $((z_j + z_j^{-1})_j)^{-\beta} \in \mathbb{R}(\mathbf{z})$, we see that $(\pm i)(((\mu(z_j))_j)^{\alpha} - ((\mu(z_j))_j)^{-\alpha}) \in \mathbb{R}(\mathbf{z})$, which completes the proof.

(k) By Lemma 4.3(d), for $x \in \mathbb{R}^n$, $(\mu(x_1), \ldots, \mu(x_n)) \in (\partial\mathbb{D})^n$. Then using the definition of $M(f)$, when $x \in \mathbb{R}^n$, $M(f)(x) = f(\mu(x_1), \ldots, \mu(x_n)) \geq 0$.

□

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1: When $q \in \mathbb{C}[\mathbf{z}^{\pm 1}]$, for $z \in (\partial\mathbb{D})^n$, $\overline{q(z)} = \sum_{\alpha} \overline{q_{\alpha}} z^{-\alpha} =: \bar{q}(z)$, where $\bar{q} \in \mathbb{C}[\mathbf{z}^{\pm 1}]$. Then for $f \in \mathbb{C}(\mathbf{z})$, $f = p/q$ in lowest terms, $f = (p\bar{q})/(q\bar{q})$, where $q\bar{q} \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ with $(q\bar{q})(z) = |q(z)|^2$ for all $z \in (\partial\mathbb{D})^n$. This means that $p\bar{q} \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ satisfies $(p\bar{q})(z) = f(z)|q(z)|^2 \geq 0$ for all $z \in (\partial\mathbb{D})^n$. Then if we can prove the result for functions in $\mathbb{C}[\mathbf{z}^{\pm 1}]$, we will have that $(p\bar{q})(z) = \sum_{j=1}^{J} |g_j(z)|^2$ for all $z \in (\partial\mathbb{D})^n$, for some $g_j \in \mathbb{C}(\mathbf{z})$ and $J \leq 2^n$, and thus $f(z) = \sum_{j=1}^{J} |(g_j/q)(z)|^2$ for all $z \in (\partial\mathbb{D})^n$, where $g_j/q \in \mathbb{C}(\mathbf{z})$, since by hypothesis $q \in \mathbb{C}(\mathbf{z})$ is nonzero.

Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ such that $f(z) \geq 0$ for all $z \in (\partial\mathbb{D})^n$. Then $0 = f(z) - \overline{f(z)} = \sum_{\alpha}(f_{\alpha} - \overline{f_{-\alpha}})z^{\alpha}$ for all $z \in (\partial\mathbb{D})^n$, so $f_{\alpha} = \overline{f_{-\alpha}}$ for all $\alpha \in \mathbb{Z}^n$. Then $f_{\alpha} = \frac{1}{2}(f_{\alpha} + \overline{f_{\alpha}}) + i\left(\frac{1}{2i}(f_{\alpha} - \overline{f_{\alpha}})\right) = \frac{1}{2}(f_{\alpha} + f_{-\alpha}) + i\left(\frac{1}{2i}(f_{\alpha} - f_{-\alpha})\right)$, so

$$\begin{aligned}
f(z) &= f_0 + \sum_{\alpha >_{\mathrm{lex}} 0} f_{\alpha} z^{\alpha} + f_{-\alpha} z^{-\alpha} \\
&= f_0 + \sum_{\alpha >_{\mathrm{lex}} 0} \frac{1}{2}(f_{\alpha} + f_{-\alpha})(z^{\alpha} + z^{-\alpha}) + i\left(\frac{1}{2i}(f_{\alpha} - f_{-\alpha})\right)(z^{\alpha} - z^{-\alpha}) \\
&= g_1 + i(g_2 - \tilde{g}_2),
\end{aligned}$$

where $g_1, g_2 \in \mathbb{R}(\mathbf{z})$, since $g_1 = f_0 + \sum_{\alpha >_{\mathrm{lex}} 0} \frac{1}{2}(f_{\alpha} + \overline{f_{\alpha}})(z^{\alpha} + z^{-\alpha})$, which is actually in

$\mathbb{R}(\mathbf{z})_{\sim}$, and $g_2 = \sum_{\alpha >_{\text{lex}} 0} \frac{1}{2i}(f_\alpha - \overline{f_\alpha})z^\alpha$, which is clearly in $\mathbb{R}(\mathbf{z})$.

Let M be the extended forward Möbius transformation, and let $\widetilde{M}$ be the extended reverse Möbius transformation. Now from Lemma 4.4(i) and (j), we see that $M(f) \in \mathbb{R}(\mathbf{z})$, since $M(g_1) \in \mathbb{R}(\mathbf{z})$, $M(i(g_2 - \tilde{g}_2)) = iM(g_2 - \tilde{g}_2) \in \mathbb{R}(\mathbf{z})$. By Lemma 4.4(k), $M(f)(x) \geq 0$ for all $x \in \mathbb{R}^n$. Then by Corollary 1.1, $M(f) = \sum_{j=1}^J r_j^2$, where $r_j \in \mathbb{R}(\mathbf{z})$ for $1 \leq j \leq J \leq 2^n$. Now $\widetilde{M}(r_j) \in \mathbb{C}(\mathbf{z})$, and by Lemma 4.4(h), for $z \in (\partial\mathbb{D})^n$, $\widetilde{M}(r_j)(z) \in \mathbb{R}$. By Lemma 4.4(b), $f = \widetilde{M}(M(f)) = \widetilde{M}(\sum_{j=1}^J r_j^2) = \sum_{j=1}^J \widetilde{M}(r_j)^2$. We just showed that $\widetilde{M}(r_j)$ are real-valued on $(\partial\mathbb{D})^n$, so $f(z) = \sum_{j=1}^J |\widetilde{M}(r_j)(z)|^2$ for all $z \in (\partial\mathbb{D})^n$. This completes the proof. □

In the following example, we show how this argument proceeds in a particular case.

Example 4.1. Consider the trigonometric polynomial $1 - \cos^2(\omega_1)\cos^2(\omega_2)$, which is nonnegative for all $\omega_1, \omega_2 \in \mathbb{R}$. This corresponds (via $z_j = e^{i\omega_j}$, $j = 1, 2$) to the Laurent polynomial $f \in \mathbb{R}[z_1^{\pm 1}, z_2^{\pm 1}]$,

$$f(z_1, z_2) := 1 - \frac{1}{16}(z_1 + z_1^{-1})^2(z_2 + z_2^{-1})^2.$$

Setting $x_j \in \mathbb{R}$ for $j = 1, 2$ so that $z_j = \mu(x_j)$, we compute $M(f)(x) = f(\mu(x_1), \mu(x_2))$:

$$M(f)(x) = 1 - \frac{4}{(x_1 + x_1^{-1})^2} \frac{4}{(x_2 + x_2^{-1})^2}.$$

For each $x_1, x_2 \in \mathbb{R}$, $(\mu(x_1), \mu(x_2))$ is a point in $(\partial\mathbb{D} \setminus \{-i\})^2$, so $M(f)(x) \geq 0$ for all $x \in \mathbb{R}^2$. By Result 1.9(a), this has an sors representation on $\mathbb{R}^2$, and indeed,

$$M(f)(x) = \left(\frac{x_1 - x_1^{-1}}{x_1 + x_1^{-1}}\right)^2 + \frac{4}{(x_1 + x_1^{-1})^2}\left(\frac{x_2 - x_2^{-1}}{x_2 + x_2^{-1}}\right)^2.$$

Applying $\widetilde{M}$ to each of these squares, we get

$$f(z) = \left(\frac{z_1 - z_1^{-1}}{2i}\right)^2 + \left(\frac{z_1 + z_1^{-1}}{2}\right)^2\left(\frac{z_2 - z_2^{-1}}{2i}\right)^2,$$

and translating these back into trigonometric polynomials, we get

$$1 - \cos^2(\omega_1)\cos^2(\omega_2) = \sin^2(\omega_1) + \cos^2(\omega_1)\sin^2(\omega_2).$$

We can verify this independently, since

$$\begin{aligned}\sin^2(\omega_1) + \cos^2(\omega_1)\sin^2(\omega_2) &= 1 - \cos^2(\omega_1) + \cos^2(\omega_1)(1 - \cos^2(\omega_2)) \\ &= 1 - \cos^2(\omega_1)\cos^2(\omega_2).\end{aligned}$$

$\square$

Example 4.2. We consider an example of a nonnegative trigonometric polynomial for which there is no sos representation, from [11, Th. 2.5]. We begin by considering $\tau(\omega) = (1-c\cdot r(\omega))a(\omega)$, where $0 < c \leq 1/3$, $r(\omega) = \sin^4(\omega_1)\sin^2(\omega_2)+\sin^2(\omega_1)\sin^4(\omega_2)+\sin^6(\omega_3)-3\sin^2(\omega_1)\sin^2(\omega_2)\sin^2(\omega_3)$ is the Motzkin polynomial evaluated at $(\sin(\omega_1), \sin(\omega_2), \sin(\omega_3))$, and we choose $a(\omega) = m(\omega_1)m(\omega_2)m(\omega_3)$, where m is a lowpass mask with accuracy number at least 8 (i.e., $D^j m(\pi) = 0$ for all $0 \leq j < 8$), and satisfies the sub-QMF condition $t(\omega) := |m(\omega)|^2 + |m(\omega+\pi)|^2 \leq 1$. Using the $\pi\mathbb{Z}^3$-periodicity of the first factor of τ, we see that

$$f(\omega) := 1 - \sum_{\gamma\in\{0,\pi\}^3} |\tau(\omega+\gamma)|^2 = 1 - (1-cr(\omega))^2 \prod_{j=1}^{3} t(\omega_j),$$

where f is known not to have an sos representation from [11]. We have

$$\begin{aligned}f(\omega) = 1 &- (1-cr(\omega))^2 + (1-cr(\omega))^2(1-t(\omega_1)) \\ &+ (1-cr(\omega))^2 t(\omega_1)(1-t(\omega_2)) + (1-cr(\omega))^2 t(\omega_1)t(\omega_2)(1-t(\omega_3)).\end{aligned}$$

The latter 3 terms are all squares, since t and $1-t$ have sos representations with a single generator by the Fejér-Riesz Lemma (see Result 1.8(a)). For the first two terms, $1-(1-cr(\omega))^2 = 2cr(\omega)(1-(c/2)r(\omega))$, the right factor of which has an sos representation by Result 1.8(d) since $r(\omega) \leq 3$, which means that $1-(c/2)r(\omega) \geq 1/2 > 0$ for all $\omega \in \mathbb{T}^3$. The remaining factor $2cr$ does not have an sos representation, but it does have an sors representation: Let $u(\omega) = \sin^2(\omega_1) + \sin^2(\omega_2)$. Then

$$\begin{aligned} r(\omega) = & \frac{\sin^2(\omega_1)\sin^2(\omega_2)(u(\omega) - 2\sin^2(\omega_3))^2(u(\omega) + \sin^2(\omega_3))}{u(\omega)^2} \\ & + \frac{(\sin^2(\omega_1) - \sin^2(\omega_2))^2 \sin^6(\omega_3)}{u(\omega)^2}, \end{aligned}$$

so we see that f has an sors representation. $\square$

4.2 Matrix Results

Now we extend this theorem to the case of matrices with trigonometric polynomial entries. We start by generalizing the following result to the case where the matrix has entries in a field $F(i)$, where F is a formally real field (see Section 1.6.2), and $i = \sqrt{-1}$.

Result 4.1 (Hillar and Nie '07 [33])**.** *Let F be a real field and let A be a symmetric matrix with entries in F. If the principal minors of A can be expressed as sums of squares in F, then A is a sum of squares of symmetric matrices with entries in F.*

Remark 4.1 (Representing elements of $F(i)$)**.** For $f \in F(i)$, we show that $f = g + ih$, with $g, h \in F$: Observe that $\mathcal{F} = \{g + ih : g, h \in F\} \subseteq F(i)$, so if $\mathcal{F}$ is a field, it must be all of $F(i)$, since $F \subset \mathcal{F}$, and $i \in \mathcal{F}$. We see that $\mathcal{F}$ is a ring, since it is clearly closed under addition, and $(g_1 + ih_1)(g_2 + ih_2) = (g_1 g_2 - h_1 h_2) + i(h_1 g_2 + g_1 h_2) \in \mathcal{F}$, so $\mathcal{F}$ is closed under multiplication. To see that $\mathcal{F}$ is a field, if $g + ih \neq 0$, then $g^2 + h^2 > 0$ in any ordering of F, and

$$(g + ih)\frac{g - ih}{g^2 + h^2} = \frac{g^2 + h^2}{g^2 + h^2} = 1,$$

where $g/(g^2 + h^2), h/(g^2 + h^2) \in F$, so every nonzero element of $\mathcal{F}$ is invertible. $\square$

For $f \in F(i)$, $f = g + ih$, we define $\bar{f} = g - ih$. From the calculation of the product of two elements of $F(i)$ above, $\overline{fg} = \bar{f}\bar{g}$, and it is clear that $\overline{f + g} = \bar{f} + \bar{g}$. If $f \in F(i)$ satisfies $f = \bar{f}$, then $g + ih = f = \bar{f} = g - ih$, so $2ih = 0$, in which case $h = 0$, so $f = g \in F$.

When A is a matrix with entries in $F(i)$, we define A^* to be the transpose of A with $\bar{\cdot}$ applied to every entry. From the properties of transpose and $\bar{\cdot}$ just shown, we see that

$(AB)^* = B^*A^*$, and clearly $(A^*)^* = A$. We say that a matrix $A \in M_n(F(i))$ is symmetric when $A = A^*$.

We now state the generalization of Result 4.1.

Theorem 4.2. *Let F be a real field, and let $A \in M_n(F(i))$ be a symmetric matrix. If the principal minors of A can be expressed as sums of squares in F, then A is a sum of squares of commuting symmetric matrices $B_k \in M_n(F(i)), 1 \leq k \leq K$.*

For a square matrix A of order n, and given $\alpha \subseteq \{1, \dots, n\}$, we denote by $A[\alpha]$ the submatrix of A with rows and columns indexed by α. Note that if $A \in M_n(F(i))$ is symmetric, then for $|\alpha| = k$, $\det(A[\alpha]) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{j=1}^{k} A_{\alpha_j, \alpha_{\sigma(j)}}$, so

$$\begin{aligned}
\overline{\det(A[\alpha])} &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{j=1}^{k} \overline{A_{\alpha_j, \alpha_{\sigma(j)}}} \\
&= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{j=1}^{k} A_{\alpha_{\sigma(j)}, \alpha_j} \\
&= \det(A[\alpha]^T) = \det(A[\alpha]).
\end{aligned}$$

Therefore, all of the principal minors of A belong to F. Thus, the assumption being made on the principal minors of A is that they are totally positive in F, and not that they are elements of F.

We need a few definitions before we can prove this theorem.

Let $\langle x, y \rangle$ be defined for all $x, y \in F(i)^n$ by y^*x, where we think of x and y as column vectors with entries in $F(i)$. Then we have the following properties, which are identical to the corresponding ones for the Euclidean inner product on $\mathbb{C}^n = \mathbb{R}(i)^n$.

Lemma 4.5 (Properties of $\langle \cdot, \cdot \rangle$)**.** *Let $x, y, z \in F(i)^n$. The following properties hold:*

(a) $\langle x, y \rangle \in F(i)$.

(b) $\langle y, x \rangle = \overline{\langle x, y \rangle}$.

(c) $\langle ax, y \rangle = a \langle x, y \rangle$ *whenever* $a \in F(i)$.

(d) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

(e) $\langle x, x\rangle \in F$.

(f) Given an ordering $\leq$ *on* F, $\langle x, x\rangle \geq 0$, *and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$.

Note that when $F = \mathbb{R}$, the ordering $\leq$ in part (f) is unique, but this is not necessarily true for other formally real fields F.

Proof. (a) $y^*x = \sum_{j=1}^n x_j\overline{y_j}$ is a sum of elements of $F(i)$, hence belongs to $F(i)$.

(b) Let $x = x_r + ix_i, y = y_r + iy_i$, with $x_r, x_i, y_r, y_i \in F^n$. Then $x^*y = (x_r^T - ix_i^T)(y_r + iy_i) = x_r^T y_r + x_i^T y_i - i(x_i^T y_r - x_r^T y_i)$, and $y^*x = (y_r^T - iy_i^T)(x_r + ix_i) = y_r^T x_r + y_i^T x_i + i(y_r^T x_i - y_i^T x_r)$. Since $u^T v = v^T u$ for $u, v \in F^n$, the result follows.

(c) $y^*(ax) = \sum_{j=1}^n ax_j\overline{y_j} = a\sum_{j=1}^n x_j\overline{y_j} = a(y^*x)$

(d) $z^*(x + y) = \sum_{j=1}^n (x_j + y_j)\overline{z_j} = \sum_{j=1}^n x_j\overline{z_j} + y_j\overline{z_j} = z^*x + z^*y$

(e) Applying (b) with $y = x$, $\overline{\langle x, x\rangle} = \langle x, x\rangle$. Thus, $\langle x, x\rangle \in F$.

(f) Using the calculation in (b) with $y = x$, $\langle x, x\rangle = x_r^T x_r + x_i^T x_i$. Each of $x_r^T x_r$ and $x_i^T x_i$ is a sum of squares of elements of F, so $\langle x, x\rangle \geq 0$, and when this equals zero, every component of x_r, x_i must be zero, which means that $x = 0$. The converse is obvious.

□

We now present the proof of Theorem 4.2, which follows in roughly 7 large steps, that we briefly sketch here to help make the structure of the argument clear: First, we show that extending F to its real closure R, for any vector $x \in R(i)^n$, $\sqrt{\langle x, x\rangle}$ is a well-defined element of R. This lets us complete the normalization step in the proof that the Gram-Schmidt orthonormalization procedure works in this setting. After this, we prove that under the given assumptions, A has eigenvalues and eigenvectors in $R(i)$, as well as a "unitary" upper triangularization, where we use Gram-Schmidt to find the appropriate "unitary" matrix. Applying $A = A^*$, we will see that this is a "unitary" diagonalization, and the eigenvalues of A actually belong to R. From this information, and using the assumption about the principle minors of A, we will obtain a wealth of information about the minimal polynomial of A:

$$q(t) = \sum_{k=0}^{s} (-1)^{s-k} a_k t^k,$$

where the a_k are all sums of squares in F (among other properties). Since $q(A) = 0$, splitting between odd and even powers of A gives

$$\left(\sum_{k \text{ odd}} a_k A^{k-1} \right) A = \left(\sum_{k \text{ even}} a_k A^k \right),$$

where the factor multiplying A (call this B) is invertible and a sum of squares. Finally,

$$A = B \left(\sum_{k \text{ even}} a_k B^{-2} A^k \right)$$

gives a sum of squares representation for A, since A and B^{-1} commute.

Now we proceed to the proof.

Proof of Theorem 4.2: Let R be the real closure of F extending the ordering on F. We see that we may define $\langle \cdot, \cdot \rangle$ on all pairs of vectors in $R(i)^n$ as we did for $F(i)$, and all of the properties of Lemma 4.5 will hold, since we could have chosen $F = R$ in the application of that lemma. Then we see that for any $x \in R(i)^n$, $\sqrt{\langle x, x \rangle} \in R$, since $\langle x, x \rangle \in R$ and $\langle x, x \rangle \geq 0$ by Lemma 4.5(e) and (f). We will now prove that the Gram-Schmidt orthonormalization procedure works in this setting just as it does in $\mathbb{C}^n$. This proof is the standard one by induction, essentially replacing all of the needed properties of the usual inner product on $\mathbb{C}^n$ with the ones of Lemma 4.5 for $\langle \cdot, \cdot \rangle$ on $R(i)^n$.

Claim: Let $v_1, \ldots, v_r$ be a collection of vectors in $R(i)^n$. We will show that there is a collection of vectors $w_1, \ldots, w_s \in R(i)^n$ with the property that for each $1 \leq m \leq r$, there is some $1 \leq k \leq m$ such that $\operatorname{span}\{v_1, \ldots, v_m\} = \operatorname{span}\{w_1, \ldots, w_k\}$, and such that for all $1 \leq j, k \leq s$, $\langle w_j, w_k \rangle = \delta(j-k)$, which is 1 when $j = k$ and 0 otherwise.

Proof of claim: If all of the vectors $v_m = 0$, then the empty set has the stated properties. Otherwise, take the first nonzero vector v_{m_1} in the list $v_1, \ldots, v_r$, and let u_1 be equal to it. Then $w_1 = u_1 / \sqrt{\langle u_1, u_1 \rangle} \in R(i)^n$, since $u_1 = v_{m_1} \in R(i)^n$, and $\sqrt{\langle u_1, u_1 \rangle} \in R$ is nonzero since v_{m_1} is nonzero. Now suppose that $w_1, \ldots, w_{k-1}$ have been defined, and there are still vectors v_m which have not been used. For each of the vectors remaining in the list $v_1, \ldots, v_r$, check whether $v_m - \sum_{\ell=1}^{k-1} \langle v_m, w_\ell \rangle w_\ell = 0$. If this is true for all of the vectors remaining in the list, return $\{w_1, \ldots, w_{k-1}\}$; otherwise, let $u_k = v_{m_k} - \sum_{\ell=1}^{k-1} \langle v_{m_k}, w_\ell \rangle w_\ell$ for the first vector v_{m_k} in the remaining list such that $u_k \neq 0$. Let $w_k = u_k / \sqrt{\langle u_k, u_k \rangle}$. As argued in the first case, $w_k \in R(i)^n$. Since r is finite, this procedure will terminate after finitely many steps, giving us a list $\{w_1, \ldots, w_s\}$. Moreover, from the construction procedure, if we are given a number $1 \leq m \leq r$, then if we let $m_0 = 0$, $m_{s+1} = r+1$, either $m_k < m < m_{k+1}$ for some k, in which case $\mathrm{span}(\{v_1, \ldots, v_m\}) = \mathrm{span}(\{w_1, \ldots, w_k\})$, or else $m = m_k$ for some $1 \leq k \leq s$, but in this case, the same equality holds.

We see that for each k, $\langle w_k, w_k \rangle = \langle u_k, u_k \rangle / \langle u_k, u_k \rangle = 1$. When $k = 1$, clearly $\langle w_j, w_\ell \rangle = \delta(j - \ell)$ for all $1 \leq j \leq k$. So suppose by way of induction that $\langle w_j, w_\ell \rangle = \delta(j - \ell)$ for all $1 \leq j, \ell \leq k-1$, where $k - 1 \geq 1$. Let j satisfy $1 \leq j \leq k-1$. Then

$$\langle w_k, w_j \rangle = \frac{\langle v_{m_k}, w_j \rangle - \sum_{\ell=1}^{k-1} \langle v_{m_k}, w_\ell \rangle \langle w_\ell, w_j \rangle}{\sqrt{\langle u_k, u_k \rangle}}.$$

Since $\langle w_\ell, w_j \rangle = \delta(j - \ell)$ whenever $1 \leq j, \ell \leq k-1$ by the induction hypothesis,

$$\langle w_k, w_j \rangle = \frac{\langle v_{m_k}, w_j \rangle - \sum_{\ell=1}^{k-1} \langle v_{m_k}, w_\ell \rangle \delta(\ell - j)}{\sqrt{\langle u_k, u_k \rangle}} = 0.$$

Applying Lemma 4.5(b), this is also true for $\langle w_j, w_k \rangle$, so by induction, $\langle w_k, w_j \rangle = \delta(k - j)$ holds for all $1 \leq k, j \leq s$. This completes the proof of the claim. $\square$

Since $A \in M_n(F(i))$, its characteristic polynomial $f_A(t) \in F(i)[t]$, and therefore splits in $R(i)$, so A has n eigenvalues (counting multiplicities) in $R(i)$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A in any prescribed order. In any field k, for $B \in M_n(k)$, $\det(B) = 0$ if and only if there is a nontrivial solution $x \in k^n$ to $Bx = 0$ [43]. Then by definition of the characteristic polynomial, $f_A(\lambda_j) = 0$ if and only if $\det(A - \lambda_j I) = 0$, which is true if and only if there is some $x \in R(i)^n$ such that $(A - \lambda_j I)x = 0$, in which case $Ax = \lambda_j x$.

The following part of the proof is adapted from Horn and Johnson's proof of Schur's Lemma [34]. If $x \in R(i)^n$ is an eigenvector of A with eigenvalue λ_1, $\hat{x} = x / \sqrt{\langle x, x \rangle} \in R(i)^n$ is also an eigenvector for this eigenvalue (note that x must be nonzero, so the denominator is nonzero by Lemma 4.5(f)), and $\langle \hat{x}, \hat{x} \rangle = 1$, so we are free to assume that x is normalized with $\langle x, x \rangle = 1$. If we apply the Gram-Schmidt procedure to the set $\{x, e_1, e_2, \ldots, e_n\}$, where e_i are the standard unit vectors, then we will arrive at a set $\{x, w_2, \ldots, w_n\} \subset R(i)^n$ (since $x \neq 0$) with the property that the matrix $U_1 = [x\, w_2\, \cdots\, w_n] \in M_n(R(i))$ satisfies $U_1^* U_1 = I$, since this is just another way of writing the orthogonality conditions $\langle w_j, w_k \rangle = \delta(j-k)$ from the claim. Moreover,

$$U_1^* A U_1 = \begin{bmatrix} x^* \\ w_2^* \\ \vdots \\ w_n^* \end{bmatrix} [Ax\, Aw_2\, \cdots\, Aw_n] = \begin{bmatrix} x^* \\ w_2^* \\ \vdots \\ w_n^* \end{bmatrix} [\lambda_1 x\, Aw_2\, \cdots\, Aw_n] = \begin{bmatrix} \lambda_1 & \star \\ 0 & A_1 \end{bmatrix}.$$

Inspecting the characteristic polynomial of both sides of the previous equation, we see that A_1 has eigenvalues $\lambda_2, \ldots, \lambda_n$, so either A_1 is upper triangular, and we have obtained an upper triangular matrix, or else we may repeat the previous procedure to find a matrix $U_2 \in M_{n-1}(R(i))$ satisfying $U_2^* U_2$ and such that $U_2^* A_1 U_2 = \begin{bmatrix} \lambda_2 & \star \\ 0 & A_2 \end{bmatrix}$, where A_2 has eigenvalues $\lambda_3, \ldots, \lambda_n$. Continuing in this fashion, and letting $V_j = [I_{j-1}] \oplus [U_j]$ for each $1 \leq j \leq n$, we have that $V_n^* V_{n-1}^* \cdots V_1^* A V_1 \cdots V_n$ is upper triangular with diagonal entries $\lambda_1, \ldots, \lambda_n$ in the given order[1].

Now we apply the symmetry $A = A^*$. From what was just shown, there is a matrix $U \in M_n(R(i))$ with $U^* U = I$ and $U^* A U = T$ is upper triangular with the eigenvalues $\lambda_1, \ldots, \lambda_n$ on the diagonal of T (in any prescribed order). But we see that $T^* = (U^* A U)^* = U^* A^* U = U^* A U = T$, so T is diagonal with elements of R on the diagonal, since $\overline{T_{j,j}} = T_{j,j}$ for all $1 \leq j \leq n$. That is, A is "unitarily" diagonalizable, and its eigenvalues belong to R.

We have supposed additionally that all of the principal minors of A are sums of squares in F. Then $E_k(A) = \sum_{|\alpha|=k} \det(A[\alpha])$ are sums of squares in F for all $1 \leq k \leq n$, so if $p \in F(i)[t]$ is the characteristic polynomial of A, from [34],

$$p(t) = \sum_{k=0}^{n} (-1)^{n-k} E_{n-k}(A) t^k \in F[t].$$

[1] In the case that we are just in $\mathbb{C}$, this is the conclusion of Schur's Lemma.

Let $\leq$ be an ordering on F, and let the ordering on R, the real closure of F, extend this ordering. Then whenever $t \in R$ is such that $t < 0$, $(-1)^n p(t) = \sum_{k=0}^{n} E_{n-k}(A)(-t)^k > 0$, since each of the terms $E_{n-k}(A)(-t)^k \geq 0$, and $E_0(A)(-t)^n = (-t)^n > 0$, since this is a power of a positive element of R. As such, all of the roots of $p(t)$ are nonnegative.

Moreover, since A is diagonalizable in $M_n(R(i))$, we see that the minimal polynomial $q(t) \in F(i)[t]$ of A has distinct linear factors, and since all of the roots of q are roots of p (which belong to R) we see that $q(t)$ must also belong to $R[t]$, which means that $q(t) \in F(i)[t] \cap R[t] = F[t]$.

From this point forward, the argument proceeds along the lines of the proof of Hillar and Nie's result in [33]. Writing

$$q(t) = \sum_{k=0}^{s} (-1)^{s-k} a_k t^k,$$

since $a_k = S_k(\lambda_1, \ldots, \lambda_s)$, where $\lambda_1, \ldots, \lambda_s \in R$, $\lambda_k \geq 0$ for all $1 \leq k \leq s$, are the distinct eigenvalues of A, and S_k is the kth elementary symmetric polynomial, we see that $a_k \geq 0$ in R for all $0 \leq k \leq s$, since it is a sum of products of nonnegative elements of R. If a_k were not an sos of elements of F, then there would be some ordering of F for which $a_k < 0$, and extending this ordering to R would result in a contradiction: thus, a_k must be an sos of elements of F for each k.

Since q has distinct linear factors, it is not divisible by t^2, so not both of a_1, a_0 may be 0. Since $a_1 = S_1(\lambda_1, \ldots, \lambda_s) = \sum_{k=1}^{s} \prod_{j \neq k} \lambda_j$, where $\lambda_j \geq 0$ in R for all $1 \leq j \leq s$, if $a_1 = 0$, then some $\lambda_j = 0$, so $(-1)^s a_0 = (-1)^s \prod_{j=1}^{s} \lambda_j = 0$, which would mean that t^2 divides q, a contradiction. Thus, $a_1 \neq 0$. We have now shown that [33, Lemma 4] can be extended to the setting where $A \in M_n(F(i))$.

Now we see that

$$q(A) = A^s - a_{s-1}A^{s-1} + a_{s-2}A^{s-2} + \cdots + (-1)^s a_0 I = 0,$$

so if s is odd, we have $(A^{s-1} + a_{s-2}A^{s-3} + \cdots + a_1 I)A = a_{s-1}A^{s-1} + \cdots + a_0 I$. The eigenvalues of $B = (A^{s-1} + a_{s-2}A^{s-3} + \cdots + a_1 I)$ are $\gamma_j = \lambda_j^{s-1} + a_{s-2}\lambda_j^{s-3} + \cdots + a_1$, where using the ordering on R, $\lambda_j \geq 0$ for all $1 \leq j \leq n$, $a_j \geq 0$ for all $1 \leq j \leq s$, and $a_1 > 0$ since it is nonzero. Then $\gamma_j > 0$ for all $1 \leq j \leq n$, so B is invertible, and is a sum of squares of symmetric matrices in $M_n(F(i))$, since $s-1, s-3, \ldots, 0$ are all even integers, $A \in M_n(F(i))$ is symmetric, and all of the a_j are sos of elements in F. Then we have

$$\begin{aligned} A &= B(a_{s-1}B^{-1}A^{s-1}B^{-1} + a_{s-3}B^{-1}A^{s-3}B^{-1} + \cdots + a_0B^{-2}) \\ &= B(a_{s-1}(B^{-1}A^{(s-1)/2})^2 + a_{s-3}(B^{-1}A^{(s-3)/2})^2 + \cdots + a_0B^{-2}) \end{aligned} \tag{4.2}$$

(using the commutativity in the second equality), which is a sum of squares of symmetric matrices in $M_n(F(i))$. It is also clear that these sums commute.

When s is even, $(a_{s-1}A^{s-2} + a_{s-3}A^{s-4} + \cdots + a_1I)A = A^s + a_{s-2}A^{s-2} + \cdots + a_0I$, and as before, the matrix $B = (a_{s-1}A^{s-2} + a_{s-3}A^{s-4} + \cdots + a_1I)$ is invertible, and is a sum of squares of symmetric matrices in $M_n(F(i))$. Thus

$$A = B((B^{-1}A^{s/2})^2 + a_{s-2}(B^{-1}A^{(s-2)/2})^2 + \cdots + a_0B^{-2})$$

is a sum of squares of commuting symmetric matrices in $M_n(F(i))$. This completes the proof. □

Example 4.3 (Number of Sos Generators Depends on Field)**.** It is worth pointing out that if we are willing to accept an sos representation with matrices in $M_n(R(i))$, then we only need one sos generator in the result above. Once we have shown that $A = UDU^*$, where D is a diagonal matrix with diagonal entries in R, and all diagonal elements $D_{j,j} \geq 0$ in the ordering on R, then letting C be a diagonal matrix with $C_{j,j} = \sqrt{D_{j,j}} \in R$, we see that $A = (UCU^*)^2$, where UCU^* is clearly symmetric. The rest of the argument shows that we have an sos representation with matrices in $M_n(F(i))$. As an application, consider the matrix $A \in M_2(\mathbb{Q}(i))$, where

$$A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \right)^2$$

is a sos representation with a single generator belonging to $M_2(\mathbb{C})$. On the other hand, we can get an sos representation with 2 matrices in $M_2(\mathbb{Q}(i))$ as follows:

$$\left(\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}\right)^2 + \left(\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}\right)^2 = \frac{1}{2}A + \frac{1}{2}A = A.$$

$\square$

The following corollary may be viewed as a generalization of Result 1.10, but slightly strengthened because of the bound on the number of sos generators. However, Example 4.3 should serve as a warning that this upper bound may dramatically overestimate the number of sos generators needed for the sos representation of a particular matrix. Following the terminology in the rest of this section, we say that $A \in M_m(\mathbb{C}(\mathbf{z}))$ is Hermitian or positive semidefinite on $\Omega \subseteq \mathbb{C}^n$ when $A(z)$ satisfies the corresponding property for all $z \in \Omega$ at which $A(z)$ is defined.

Corollary 4.1. *Let $A \in M_m(\mathbb{C}(\mathbf{x}))$ such that A is Hermitian and positive semidefinite on $\mathbb{R}^n$. Then A is a sum of squares on $\mathbb{R}^n$ of commuting matrices $B_k \in M_m(\mathbb{C}(\mathbf{x})), 1 \leq k \leq K$, which are Hermitian on $\mathbb{R}^n$, and $K \leq 2^{n-1}m(2^{n-1}m+1)$. In particular, $A(x) = \sum_{k=1}^{K} B_k(x)^2 = \sum_{k=1}^{K} B_k(x)^* B_k(x)$ for all $x \in \mathbb{R}^n$, where $*$ here denotes the ordinary conjugate transpose on $M_n(\mathbb{C})$.*

Proof. For $A \in M_m(\mathbb{C}(\mathbf{x}))$, since we are considering evaluations $x \in \mathbb{R}^n$, the involution $*$ on $M_m(\mathbb{C}(\mathbf{x})) = M_m(\mathbb{R}(\mathbf{x})(i))$ as defined in this section agrees with the ordinary conjugate transpose, which is to say that $A^*(x)$, the evaluation of A^* in the sense of $M_m(\mathbb{C}(\mathbf{x}))$ at the point $x \in \mathbb{R}^n$, is just equal to $(A(x))^*$, the ordinary conjugate transpose on $M_m(\mathbb{C})$. Then our Hermitian assumption here implies that $A = A^*$ as an element of $M_m(\mathbb{C}(\mathbf{x}))$. The remark after the statement of Theorem 4.2 proved that all of the principal minors of A belong to $\mathbb{R}(\mathbf{x})$, and the positive semidefiniteness assumption on A is equivalent to assuming that all of these principal minors are nonnegative for all $x \in \mathbb{R}^n$ at which they are defined. By Corollary 1.1, this means that each of the principal minors of A is a sum of squares of functions in $\mathbb{R}(\mathbf{x})$. Now we apply Theorem 4.2, which yields the conclusion immediately.

In fact, since Corollary 1.1 tells us that the a_k in the proof of Theorem 4.2 all have at most 2^n sos generators in $\mathbb{R}(\mathbf{x})$, and $s \leq m$ in that proof, we see that in Equation 4.2, B has at most $1+2^{n-1}(m-1)$ generators, and the right factor has at most $2^{n-1}(m+1)$ sos generators, which gives a total of $4^{n-1}(m^2-1)+2^{n-1}(m+1) = 4^{n-1}m^2+2^{n-1}m+2^{n-1}(1-2^n) \leq 2^{n-1}m(2^{n-1}m+1)$ sos generators for A. When m is even, B has at most $2^{n-1}m$ generators, and the other factor has at most $1+2^{n-1}m$ generators, which gives $2^{n-1}m(2^{n-1}m+1)$ total generators. □

The following corollary combines the ideas of the proof of Theorem 4.1 with Theorem 4.2.

Corollary 4.2 (Matrix Version)**.** *Let $A \in M_m(\mathbb{C}(\mathbf{z}))$ such that A is Hermitian and positive semidefinite on $(\partial\mathbb{D})^n$. Then A is a sum of squares on $(\partial\mathbb{D})^n$ of commuting matrices $B_k \in M_m(\mathbb{C}(\mathbf{z})), 1 \leq k \leq K$, which are symmetric on $(\partial\mathbb{D})^n$, and $K \leq 2^{n-1}m(2^{n-1}m+1)$. In particular, $A(z) = \sum_{k=1}^K B_k(z)^2 = \sum_{k=1}^K B_k(z)^* B_k(z)$ for all $z \in (\partial\mathbb{D})^n$, where $*$ here denotes the ordinary conjugate transpose on $M_n(\mathbb{C})$.*

Proof. Let $\mu, \tilde{\mu}$ be the forward and reverse Möbius transformations, and let $M, \widetilde{M}$ be the extended forward and reverse Möbius transformations, respectively. We see that the matrix $M(A) := [M(A_{i,j})]_{i,j=1}^m$ has entries in $\mathbb{C}(\mathbf{z})$, by Lemma 4.4(a). By Lemma 4.3(d), for $x \in \mathbb{R}^n$, $(\mu(x_j))_j \in (\partial\mathbb{D})^n$, so $M(A)(x) = A(\mu(x_1), \ldots, \mu(x_n))$ is Hermitian and positive semidefinite where it is defined by assumption. Now by Result 4.1, $M(A) = \sum_{k=1}^K B_k^2$, where B_k are commuting matrices with entries in $\mathbb{C}(\mathbf{x})$, and are Hermitian for all $x \in \mathbb{R}^n$ where they are defined. Then by Lemma 4.4(b), $A = \widetilde{M}(M(A)) = \widetilde{M}(\sum_{k=1}^K B_k^2) = \sum_{k=1}^K \widetilde{M}(B_k)^2$, where $K \leq 2^{n-1}m(2^{n-1}m+1)$. Clearly $\widetilde{M}(B_k)$ has entries in $\mathbb{C}(\mathbf{z})$, and for any $z \in (\partial\mathbb{D} \setminus \{-i\})^n$ at which it is defined, $\widetilde{M}(B_k)(z) = B_k(\tilde{\mu}(z_1), \ldots, \tilde{\mu}(z_n))$ is Hermitian, since by Lemma 4.3(e), for such z, $(\tilde{\mu}(z_j))_j \in \mathbb{R}^n$. Finally, we observe that $\widetilde{M}(B_j)\widetilde{M}(B_k) = \widetilde{M}(B_j B_k) = \widetilde{M}(B_k B_j) = \widetilde{M}(B_k)\widetilde{M}(B_j)$, so the commutativity of the sos generators is preserved. □

4.3 Summary

In this chapter, we proved several results about the existence of sums of rational squares representations: first, we showed that these existed for nonnegative trigonometric polynomials in any number of variables. Then, we generalized the results of [33] to show that for the appropriate notion of positive semidefiniteness, matrices with entries in a field $F(i)$, where F is a formally real field, have sums of squares representations, where the matrices generating this representation have entries in the same field. We used this result to show that for matrices with entries in $\mathbb{C}(\mathbf{z})$, positive semidefiniteness on $\mathbb{R}^n$ or $(\partial\mathbb{D})^n$ is sufficient for the existence of sums of squares representations on those sets with generators having entries also in $\mathbb{C}(\mathbf{z})$.

In the next chapter, we will see how some of these results may be applied in the context of wavelet construction with the oblique extension principle.

Chapter 5

Oblique sub-QMF Condition

5.1 OEP Tight Wavelet Frames from Sors Representations

One downside to UEP-based constructions, is that they may result in highpass masks having suboptimal vanishing moments, and the OEP (see Section 1.5.2) seeks to address this. In [42], a strong condition on the vmr function and lowpass mask was found which results in tight wavelet frames with maximum vanishing moments, and may be viewed as an "oblique QMF condition." This result is described in more detail in Section 1.7.3. In this chapter, we consider weakening this condition, resulting in what we call the "oblique sub-QMF condition" on the vmr function S and lowpass mask τ, where we also transition to the case of a general dilation matrix $\mathcal{M}$:

$$f(S, \tau; \omega) = \frac{1}{S(2\omega)} - \sum_{\gamma \in \Gamma^*} \frac{|\tau(\omega + \gamma)|^2}{S(\omega + \gamma)} \geq 0 \text{ for all } \omega \in \mathbb{T}^n.$$

In keeping with the ordinary sub-QMF and QMF conditions of Equation 1.5, when equality holds for all $\omega \in \mathbb{T}^n$, we call this the oblique QMF condition. We will show that this is in fact equivalent to the existence of wavelet masks satisfying the OEP conditions for the given S and τ, under the condition that S is a rational trigonometric polynomial. Moreover, using rational trigonometric polynomial highpass masks, we are able to eliminate the assumption about the existence of a sum of squares representation for $f(S, \tau; \cdot)$, because we have proved that nonnegative rational trigonometric polynomials have sors representations in Theorem 4.1. This stands in contrast to the UEP constructions we have considered, since in [11, Theorem 2.5], they discuss a lowpass mask τ for which there is no sos representation for $f(\tau;\cdot)$, meaning that a UEP construction with trigonometric polynomial wavelet masks is impossible. We consider this same lowpass mask in Example 4.2, showing why it has an sors representation more explicitly than by simply applying Theorem 4.1, and we will extend this in Example 5.1, finding rational trigonometric polynomial highpass masks satisfying the UEP conditions with τ, which generate a tight wavelet frame.

We will also prove a few analytical results, which show that our construction actually results in masks generating a tight wavelet frame for $L^2(\mathbb{R}^n)$. In Theorem 5.2, we do this under the assumption that S is a rational trigonometric polynomial, which is the setting under which we are able to guarantee the existence of rational trigonometric polynomial highpass masks satisfying the OEP conditions with τ and S, provided that these satisfy the oblique sub-QMF condition. We then consider the setting where we know that rational trigonometric polynomial masks $\{q_\ell\}$ satisfy the OEP conditions with τ, but S may not be a rational trigonometric polynomial. It turns out that under some conditions on S, the lowpass mask τ still satisfies the oblique sub-QMF condition with S almost everywhere, and this allows us to show that the generated wavelet system is a tight wavelet frame for $L^2(\mathbb{R}^n)$. This program is carried out in Section 5.1.2.

In Section 5.2, we discuss another interpretation of the results in this chapter, where we consider scaling an oblique version of the Laplacian pyramid matrix [8]. This interpretation lets us clearly demonstrate how different factorization assumptions on a trigonometric polynomial related to $f(S,\tau;\omega)$, namely

$$S(\mathcal{M}^T\omega) + S(\mathcal{M}^T\omega)^2 f(S,\tau;\omega),$$

lead to different filter bank constructions. In particular, we have constructions with just a modified lowpass mask, which are analogous to those of [37]; the original lowpass mask, and a collection of highpass masks corresponding to sums of squares generators for $f(S,\tau;\omega)$, which are analogous to those in [11, 42]; and constructions which combine these two ideas.

These new results allow us to construct tight wavelet frames with maximum vanishing moments in a wide variety of new settings. In Section 5.3, we focus on the case of box spline lowpass masks, and demonstrate that if a number of univariate trigonometric polynomials equal to the number of distinct directions in the box spline can be constructed satisfying certain properties, we can find a vmr function leading to a tight wavelet frame with nearly maximum vanishing moments. The reason why one might consider doing this is that it allows for a much simpler form of S than ones that achieve maximum vanishing moments, as discussed in [42]. Being able to trade off different criteria in this way while still obtaining a tight wavelet frame highlights the flexibility of our result. Our construction in this case is similar in some ways to the one in [12], and when the necessary univariate trigonometric polynomials exist, we have the same number of highpass masks as appear there. However, their method relies on the UEP, and results in tight wavelet frames with one vanishing moment.

Most of the results in this chapter first appeared in [36].

5.1.1 Construction

Given a trigonometric polynomial lowpass mask τ and rational trigonometric polynomial vmr function S, we now present the oblique sub-QMF condition, which is a condition on this pair which guarantees the existence of rational trigonometric polynomial highpass masks q_ℓ satisfying the OEP conditions (specifically (ii), in Result 1.7), and we show how to find these masks constructively. This condition might be thought of as an oblique extension of the well-known sub-QMF condition, and indeed when $S \equiv 1$, the "oblique sub-QMF condition" reduces to the sub-QMF condition, which is necessary for constructing a tight wavelet frame with the UEP (see Section 1.5.1). Analogously, the OEP conditions will necessitate our oblique version. We begin by stating the main theorems, which are split into an algebraic part and an analytical part. The proofs of these theorems then follow, along with a few requisite definitions and lemmata.

Theorem 5.1 (Oblique sub-QMF Condition is equivalent to OEP conditions)**.** *Let S be a nonzero rational trigonometric polynomial which is nonnegative on $\mathbb{T}^n$, and let τ be a trigonometric polynomial lowpass mask. The following are equivalent:*

(A) The Oblique sub-QMF condition holds:

$$\sum_{\gamma\in\Gamma^*}\frac{|\tau(\omega+\gamma)|^2}{S(\omega+\gamma)} \le \frac{1}{S(\mathcal{M}^T\omega)} \quad \textit{for all } \omega\in\mathbb{T}^n \textit{ at which both sides are defined,} \tag{5.1}$$

(B) There exist rational trigonometric polynomials q_ℓ such that for all $\gamma\in\Gamma^*$ and $\omega\in\mathbb{T}^n$ at which both sides are defined:

$$S(\mathcal{M}^T\omega)\tau(\omega)\overline{\tau(\omega+\gamma)} + \sum_{\ell=1}^{r} q_\ell(\omega)\overline{q_\ell(\omega+\gamma)} = \begin{cases} S(\omega) & \textit{if } \gamma=0 \\ 0 & \textit{otherwise.}\end{cases} \tag{5.2}$$

Moreover, provided that either (A) or (B) holds, there exist a potentially different set of rational trigonometric polynomials $q_\ell, 1\le\ell\le r$ such that for all $\gamma\in\Gamma^*$ and $\omega\in\mathbb{T}^n$ at which both sides are defined, Equation (5.2) holds, where $r\le 2^n(1+\mathcal{Q})$.

As we did for the sub-QMF condition, we will define

$$f(S,\tau;\omega) = \frac{1}{S(\mathcal{M}^T\omega)} - \sum_{\gamma\in\Gamma^*}\frac{|\tau(\omega+\gamma)|^2}{S(\omega+\gamma)},$$

the nonnegativity of which for all $\omega\in\mathbb{T}^n$ where it is defined is equivalent to the oblique sub-QMF condition (5.1).

The next theorem combines these conditions with an additional assumption on the rational trigonometric polynomial S, guaranteeing that these masks generate a TWF. This might be seen as an extension of [30, Lemma 2.1].

Theorem 5.2 (Oblique sub-QMF implies the existence of a TWF)**.** *Assume the setting of Theorem 5.1. Suppose, in addition to satisfying one of the above conditions, that S is continuous at 0 with $S(0)=1$, and that S and $1/S$ belong to $L^\infty(\mathbb{T}^n)$. Then the wavelet system defined by the combined MRA mask $(\tau, q_1,\ldots,q_r)$ is a tight wavelet frame.*

The assumptions that $S, 1/S\in L^\infty(\mathbb{T}^n)$ guarantee that S and $1/S$ are pole-free, so in this setting, Equations (5.1) and (5.2) hold for all $\omega\in\mathbb{T}^n$. Now, we turn to the proofs of these theorems. We start by making a remark on the number of $\mathcal{G}$-invariant sors generators for a

$\mathcal{G}$-invariant nonnegative rational trigonometric polynomial, which requires the information gained from Theorem 4.1.

Remark 5.1 (Number of $\mathcal{G}$-invariant Sors Generators)**.** Since we include a bound on the number of highpass masks in Theorem 5.1, this will require us to bound the number of sors generators for certain nonnegative trigonometric polynomials, which is clearly possible from the bound in Theorem 4.1. In particular, when f is $\mathcal{G}$-invariant, we showed in Lemma 1.2(b) that f is an so(r)s if and only if it is a $\mathcal{G}$-invariant so(r)s.

From the proof of Lemma 1.2(b), it may appear that the number of sors generators will increase by a factor of $\mathcal{Q}$ when we require them to be $\mathcal{G}$-invariant. However, if f is $\mathcal{G}$-invariant and an sors, then from Lemma 1.1, $f = g(\mathcal{M}^T\cdot)$, where g is certainly nonnegative. By Theorem 4.1, g has an sors $g = \sum_{j=1}^{J} |g_j|^2$, where $J \leq 2^n$, which means that $f = \sum_{j=1}^{J} |g_j(\mathcal{M}^T\cdot)|^2$ with J satisfying the same bound. On the other hand, when we want a $\mathcal{G}$-invariant *sos* representation, this argument fails because Theorem 4.1 may introduce denominators. As such, a $\mathcal{G}$-invariant sos representation may require at most $\mathcal{Q}$ times as many generators as the original sos, since this is the bound we get from the representation given in the proof of Lemma 1.2(b). $\square$

The following lemma combines Theorem 4.1 with an idea from the proof of [42, Theorem 6.1].

Lemma 5.1. *Suppose S is a nonzero rational trigonometric polynomial such that $S(\omega) \geq 0$ for all $\omega \in \mathbb{T}^n$ where it is defined. Let $\Sigma(\omega) = \operatorname{diag}(S(\omega+\gamma))_{\gamma\in\Gamma^*}$ (for some ordering of Γ^*). Then $\Sigma^{-1}(\omega) = A(\omega)A(\omega)^*$ for all $\omega \in \mathbb{T}^n$ where $\Sigma^{-1}(\omega)$ is defined, and $A(\omega)$ is a $\mathcal{Q} \times M$ matrix with rational trigonometric polynomial entries such that each column of $A(\omega)$ is a $\mathcal{G}$-vector, with $M \leq 2^n\mathcal{Q}$.*

Proof. By Theorem 4.1, there are rational trigonometric polynomials s_j such that $1/S(\omega) = \sum_{j=1}^{J} |s_j(\omega)|^2$ for all $\omega \in \mathbb{T}^n$ where $1/S(\omega)$ is defined, and $J \leq 2^n$. Let

$$\mathbf{s}(\omega) = [s_1(\omega), s_2(\omega), \ldots, s_J(\omega)],$$

and

$$A(\omega) = \mathcal{Q}^{-1/2}[e^{i\nu\cdot(\omega+\gamma)}\mathbf{s}(\omega+\gamma)]_{\gamma\in\Gamma^*,(\nu,j)\in\Gamma\times\{1,\ldots,J\}},$$

which is a $\mathcal{Q}\times(\mathcal{Q}J)$ matrix with rational trigonometric polynomial entries. Moreover, given $(\nu,j)\in\Gamma\times\{1,\ldots,J\}$, if we let $a_{(\nu,j)}(\omega) = \mathcal{Q}^{-1/2}e^{i\nu\cdot\omega}s_j(\omega)$, we see that $A(\omega)_{\gamma,(\nu,j)} = a_{(\nu,j)}(\omega+\gamma)$, so $A(\omega)$ has $\mathcal{G}$-vector columns. Then for all $\gamma,\gamma'\in\Gamma^*$, letting $\delta : 2\pi\mathbb{Z}^n \to \{0,1\}$ always take value zero except $\delta(0)=1$,

$$\begin{aligned}(A(\omega)A(\omega)^*)_{\gamma,\gamma'} &= \mathcal{Q}^{-1}\sum_{j=1}^{J} s_j(\omega+\gamma)\overline{s_j(\omega+\gamma')}\sum_{\nu\in\Gamma} e^{i\nu\cdot(\gamma-\gamma')}\\ &= \delta(\gamma-\gamma')\sum_{j=1}^{J}|s_j(\omega+\gamma)|^2\\ &= \delta(\gamma-\gamma')\frac{1}{S(\omega+\gamma)} = \Sigma^{-1}(\omega)_{\gamma,\gamma'}.\end{aligned}$$

This clearly holds wherever $\Sigma^{-1}(\omega)$ is defined. □

Another approach to the previous lemma might have applied Corollary 4.2 to obtain a representation of $\Sigma^{-1}(\omega) = \sum_{k=1}^{K} B_k(\omega)^2$, with symmetric matrices B_k having rational trigonometric polynomial entries. This yields $\Sigma^{-1}(\omega) = A(\omega)A(\omega)^*$ for $A(\omega) = [B_1(\omega)\ B_2(\omega)\cdots B_K(\omega)]$, but would not necessarily give us a matrix $A(\omega)$ with $\mathcal{G}$-vector columns, which is why we construct $A(\omega)$ explicitly in the proof above. Also, the bound from the theorem on the number of columns for $A(\omega)$ from this approach would be $\mathcal{Q}K \le 2^{n-1}\mathcal{Q}^2(2^{n-1}\mathcal{Q}+1)$, which is clearly inferior. We are able to do better in this case because of the diagonal structure of $\Sigma^{-1}(\omega)$.

Remark 5.2. If we let $H(\omega) = [\tau^\gamma(\omega)]_{\gamma\in\Gamma^*}$, and $Q(\omega) = [q_\ell^\gamma(\omega)]_{\gamma\in\Gamma^*,\ell\in\{1,\ldots,r\}}$, then we may rewrite the OEP conditions in Equation (5.2) as

$$[H(\omega)\ Q(\omega)]\begin{bmatrix} S(\mathcal{M}^T\omega) & 0\\ 0 & I\end{bmatrix}\begin{bmatrix} H(\omega)^*\\ Q(\omega)^*\end{bmatrix} = \Sigma(\omega), \tag{5.3}$$

where $\Sigma(\omega) = \mathrm{diag}(S(\omega+\gamma))_{\gamma\in\Gamma^*}$ as above. We will make use of this form of the OEP conditions in our proof of Theorem 5.1, which follows below. □

Proof of Theorem 5.1: To avoid excessive verbiage, throughout this proof, all (in)equalities should be taken to hold for all $\omega \in \mathbb{T}^n$ where both sides are defined, which because we are considering a finite sums of rational trigonometric polynomials, is an open, dense set with full measure.

(i) Proof that A implies B: Suppose Statement A of the theorem. We observe that by Theorem 4.1 and Lemma 1.2(b),

$$\frac{1}{S(\mathcal{M}^T\omega)} - \sum_{\gamma\in\Gamma^*} \frac{|\tau(\omega+\gamma)|^2}{S(\omega+\gamma)} = \sum_{j=1}^{J} |g_j(\mathcal{M}^T\omega)|^2, \tag{5.4}$$

where g_j are rational trigonometric polynomials, since the left hand side is nonnegative by Equation (5.1) and is clearly $\mathcal{G}$-invariant. Moreover, from Remark 5.1, J in Equation 5.4 is no greater than 2^n. Let $H(\omega) = [\tau(\omega+\gamma)]_{\gamma\in\Gamma^*}$ be a column vector, and let $G(\omega) = [g_j(\omega)]_{j=1}^J$. Recall that $\Sigma(\omega) = \operatorname{diag}(S(\omega+\gamma))_{\gamma\in\Gamma^*}$. Using block matrix notation, we see that

$$\begin{aligned}
&[H(\omega)\ S(\mathcal{M}^T\omega)H(\omega)G(\mathcal{M}^T\omega)^*\ \Sigma(\omega) - S(\mathcal{M}^T\omega)H(\omega)H(\omega)^*] \\
&\quad\times \begin{bmatrix} S(\mathcal{M}^T\omega) & 0 & 0 \\ 0 & I_J & 0 \\ 0 & 0 & \Sigma^{-1}(\omega) \end{bmatrix} \begin{bmatrix} H(\omega)^* \\ S(\mathcal{M}^T\omega)G(\mathcal{M}^T\omega)H(\omega)^* \\ \Sigma(\omega) - S(\mathcal{M}^T\omega)H(\omega)H(\omega)^* \end{bmatrix} \\
&= S(\mathcal{M}^T\omega)H(\omega)H(\omega)^* + \left[S(\mathcal{M}^T\omega)^2 G(\mathcal{M}^T\omega)^* G(\mathcal{M}^T\omega)\right] H(\omega)H(\omega)^* \\
&\quad + \Sigma(\omega) - \left[2S(\mathcal{M}^T\omega)\right] H(\omega)H(\omega)^* + \left[S(\mathcal{M}^T\omega)^2 H(\omega)^*\Sigma^{-1}(\omega)H(\omega)\right] H(\omega)H(\omega)^* \\
&= \Sigma(\omega) - S(\mathcal{M}^T\omega)H(\omega)H(\omega)^* \\
&\quad + S(\mathcal{M}^T\omega)^2 \left[G(\mathcal{M}^T\omega)^*G(\mathcal{M}^T\omega) + H(\omega)^*\Sigma^{-1}(\omega)H(\omega)\right] H(\omega)H(\omega)^* \\
&= \Sigma(\omega),
\end{aligned} \tag{5.5}$$

where the last equation follows by Equation (5.4). Now we use Lemma 5.1 to see that $\Sigma(\omega)^{-1} = A(\omega)A(\omega)^*$, where the M columns of A are $\mathcal{G}$-vectors, and $M \le 2^n\mathcal{Q}$. Then rewriting this matrix product, Equation (5.5) might be written

$$\begin{aligned}&[H(\omega)\ S(\mathcal{M}^T\omega)H(\omega)G(\mathcal{M}^T\omega)^*\ (\Sigma(\omega)-S(\mathcal{M}^T\omega)H(\omega)H(\omega)^*)A(\omega)]\\ &\times\begin{bmatrix} S(\mathcal{M}^T\omega) & 0\\ 0 & I_{J+M}\end{bmatrix}\begin{bmatrix} H(\omega)^*\\ S(\mathcal{M}^T\omega)G(\mathcal{M}^T\omega)H(\omega)^*\\ A(\omega)^*(\Sigma(\omega)-S(\mathcal{M}^T\omega)H(\omega)H(\omega)^*)\end{bmatrix}=\Sigma(\omega).\end{aligned}$$

Supposing that the columns of the matrix in the first row are $\mathcal{G}$-vectors, this is just another way of writing the OEP conditions for the masks generating those columns, as in Remark 5.2. Now we show that these columns are indeed $\mathcal{G}$-vectors. $H(\omega)$ is a $\mathcal{G}$-vector by definition, and the columns of $S(\mathcal{M}^T\omega)H(\omega)G(\mathcal{M}^T\omega)^*$ are also, since $H(\omega)$ is and the other factors are $\mathcal{G}$-invariant. Observing that for any rational trigonometric polynomial g,

$$\begin{aligned}&(\Sigma(\omega)-S(\mathcal{M}^T\omega)H(\omega)H(\omega)^*)[g(\omega+\gamma)]_{\gamma\in\Gamma^*}\\ &\quad=\left[S(\omega+\gamma)g(\omega+\gamma)-S(\mathcal{M}^T\omega)\tau(\omega+\gamma)\sum_{\gamma'\in\Gamma^*}\overline{\tau(\omega+\gamma')}g(\omega+\gamma')\right]_{\gamma\in\Gamma^*},\end{aligned}$$

we see that the columns of $(\Sigma(\omega)-S(\mathcal{M}^T\omega)H(\omega)H(\omega)^*)A(\omega)$ are $\mathcal{G}$-vectors, since the columns of $A(\omega)$ are. Then writing out the masks generating the columns of these two matrices, the following rational trigonometric polynomials satisfy Equation (5.2) with τ:

$$\begin{aligned}q_{1,j}(\omega)&=S(\mathcal{M}^T\omega)\tau(\omega)\overline{g_j(\mathcal{M}^T\omega)} && 1\le j\le J\le 2^n,\\ q_{2,m}(\omega)&=S(\omega)a_m(\omega)-S(\mathcal{M}^T\omega)\tau(\omega)\sum_{\gamma\in\Gamma^*}\overline{\tau(\omega+\gamma)}a_m(\omega+\gamma) && 1\le m\le M\le 2^n\mathcal{Q},\end{aligned}$$

where the mth column of $A(\omega)$ is a $\mathcal{G}$-vector for the rational trigonometric polynomial $a_m(\omega)$, $1\le m\le M$. This construction also gives the bound $r\le 2^n(1+\mathcal{Q})$.

(ii) Proof that B implies A: Suppose Statement B of the theorem. Then rearranging Equation (5.3),

$$\Sigma(\omega)-S(\mathcal{M}^T\omega)H(\omega)H(\omega)^*=Q(\omega)Q(\omega)^*,$$

where $Q(\omega)$ is as in Remark 5.2. Taking the determinant on both sides of the previous equation, the left hand gives

$$\begin{aligned}
&\det(\Sigma(\omega) - S(\mathcal{M}^T\omega)H(\omega)H(\omega)^*) \\
&\quad = \det(\Sigma^{1/2}(\omega)(I - S(\mathcal{M}^T\omega)\Sigma^{-1/2}(\omega)H(\omega)H(\omega)^*\Sigma^{-1/2}(\omega))\Sigma^{1/2}(\omega)) \\
&\quad = \det(\Sigma(\omega))(1 - S(\mathcal{M}^T\omega)H(\omega)^*\Sigma^{-1}(\omega)H(\omega)).
\end{aligned}$$

Since $Q(\omega)Q(\omega)^*$ is positive semidefinite for all $\omega \in \mathbb{T}^n$ where it is defined, its determinant is nonnegative on this set. S is nonnegative everywhere it is defined, so $1/\det(\Sigma(\omega)) \geq 0$, and $1/S(\mathcal{M}^T\omega) \geq 0$. Then Equation (5.1) follows, since

$$\frac{1}{S(\mathcal{M}^T\omega)} - \sum_{\gamma\in\Gamma^*} \frac{|\tau(\omega+\gamma)|^2}{S(\omega+\gamma)} = \frac{\det(Q(\omega)Q(\omega)^*)}{\det(\Sigma(\omega))S(\mathcal{M}^T\omega)}.$$

□

Now we turn to the analytical part of the construction, with the proof of Theorem 5.2. We will see that the additional conditions on the vmr function S are used in order to guarantee the ess. boundedness of the constructed highpass masks, as well as that of $[\hat{\phi}, \hat{\phi}]$, as required in Result 1.7(a) and (c).

Proof of Theorem 5.2: We seek to apply Result 1.7, so we will check its conditions in turn. We have assumed that $S \in L^\infty(\mathbb{T}^n)$ is nonnegative and continuous at the origin with $S(0) = 1$, so (i) holds. Moreover, from Equation (5.2), the OEP conditions hold everywhere both sides are defined, which because of the conditions on S, is all of $\mathbb{T}^n$. As such, (ii) holds. Since τ is a trigonometric polynomial, it is continuous and therefore bounded, which means that $\tau \in L^\infty(\mathbb{T}^n)$. Rearranging Equation (5.2) and looking at the case $\gamma = 0$, we see that $\sum_{\ell=1}^r |q_\ell(\omega)|^2 = S(\omega) - S(\mathcal{M}^T\omega)|\tau(\omega)|^2$, so the ess. boundedness of the right hand side implies the ess. boundedness of q_ℓ for all $1 \leq \ell \leq r$, which proves (a). Now because τ has is a trigonometric polynomial, $\hat{\phi}$ is entire, by Result 1.2, so it is continuous. Evaluating the formula $\hat{\phi}(\omega) = \prod_{j=1}^\infty \tau((\mathcal{M}^T)^{-j}\omega)$ at $\omega = 0$, the lowpass condition gives $\hat{\phi}(0) = 1$, so (b) holds. Then it remains to show (c).

We argue as in the proof of the first half of [30, Lemma 2.1]. Let

$$f_0(\omega) := \chi_{[-\pi,\pi)^n}(\omega)(S(\omega))^{-1/2}, \text{ and for all } j \geq 1, \text{ let} \tag{5.6}$$
$$f_j(\omega) := \tau(\mathcal{M}^{-T}\omega)f_{j-1}(\mathcal{M}^{-T}\omega) = \chi_{(\mathcal{M}^T)^j[-\pi,\pi)^n}(\omega)(S((\mathcal{M}^{-T})^j\omega))^{-1/2}\prod_{\ell=1}^{j}\tau((\mathcal{M}^{-T})^\ell\omega).$$

We now prove by induction that $[f_j, f_j](\omega) = \sum_{k\in\mathbb{Z}^n}|f_j(\omega+2\pi k)|^2 \leq 1/S(\omega)$. Clearly, $[f_0, f_0](\omega) = 1/S(\omega)$, so suppose by way of induction that for some $j-1 \geq 0$, $[f_{j-1}, f_{j-1}](\omega) \leq 1/S(\omega)$ for all $\omega \in \mathbb{R}^n$. Then computing and applying the induction hypothesis,

$$\begin{aligned}
[f_j, f_j](\omega) &= \sum_{k\in\mathbb{Z}^n}|\tau(\mathcal{M}^{-T}(\omega+2\pi k))|^2|f_{j-1}(\mathcal{M}^{-T}(\omega+2\pi k))|^2 \\
&= \sum_{\gamma\in\Gamma^*}\sum_{k\in\mathbb{Z}^n}|\tau(\mathcal{M}^{-T}\omega+\gamma+2\pi k)|^2|f_{j-1}(\mathcal{M}^{-T}\omega+\gamma+2\pi k)|^2 \\
&= \sum_{\gamma\in\Gamma^*}|\tau(\mathcal{M}^{-T}\omega+\gamma)|^2\sum_{k\in\mathbb{Z}^n}|f_{j-1}(\mathcal{M}^{-T}\omega+\gamma+2\pi k)|^2 \\
&= \sum_{\gamma\in\Gamma^*}|\tau(\mathcal{M}^{-T}\omega+\gamma)|^2[f_{j-1}, f_{j-1}](\mathcal{M}^{-R}\omega+\gamma) && (5.7) \\
&\leq \sum_{\gamma\in\Gamma^*}|\tau(\mathcal{M}^{-T}\omega+\gamma)|^2\frac{1}{S(\mathcal{M}^{-T}\omega+\gamma)} && (5.8) \\
&\leq \frac{1}{S(\mathcal{M}^T(\mathcal{M}^{-T}\omega))} = \frac{1}{S(\omega)}, && (5.9)
\end{aligned}$$

where we applied Equation (5.1), which holds for all $\omega \in \mathbb{T}^n$ because of the assumptions on S, to obtain the last inequality. Recall that $\hat{\phi}(\omega) := \prod_{j=1}^{\infty}\tau((\mathcal{M}^{-T})^j\omega)$ for all $\omega \in \mathbb{R}^n$. Then as $j \to \infty$, using the continuity of S at 0, $f_j(\omega) \to \hat{\phi}(\omega)$. Applying Fatou's Lemma with the counting measure, since $|f_j(\omega)|^2 \geq 0$ for all $\omega \in \mathbb{R}^n$ and $j \geq 0$, we see that

$$\begin{aligned}
[\hat{\phi}, \hat{\phi}](\omega) &= \sum_{k\in\mathbb{Z}^n}|\hat{\phi}(\omega+2\pi k)|^2 = \sum_{k\in\mathbb{Z}^n}\lim_{j\to\infty}|f_j(\omega+2\pi k)|^2 \\
&\leq \liminf_{j\to\infty}\sum_{k\in\mathbb{Z}^n}|f_j(\omega+2\pi k)|^2 \leq \frac{1}{S(\omega)} \text{ for all } \omega \in \mathbb{T}^n,
\end{aligned}$$

since $[f_j, f_j](\omega) \leq 1/S(\omega)$ for all $j \geq 1$, as we just proved. Now applying the ess. boundedness assumption on $1/S$ yields (c), which completes the proof. □

Remark 5.3. It is easy to see that under the weaker assumption that $1/S$ is integrable over $[-\pi,\pi)^n$, we may not have that all of the masks S, q_ℓ are ess. bounded, so that (i) or (a) may not hold, but the argument for (c) shows that

$$\|\hat{\phi}\|_2^2 = \int_{[-\pi,\pi)^n} [\hat{\phi},\hat{\phi}](\omega)\mathrm{d}\omega \leq \int_{[-\pi,\pi)^n} \frac{1}{S(\omega)}\mathrm{d}\omega < +\infty,$$

which shows that $\hat{\phi}$, and therefore also ϕ, belongs to $L^2(\mathbb{R}^n)$.

Combining these two theorems yields the following corollary, which applies in the setting that τ satisfies the ordinary sub-QMF condition.

Corollary 5.1. *Let τ be a trigonometric polynomial lowpass mask. The following are equivalent:*

(A) The sub-QMF condition holds:

$$\sum_{\gamma\in\Gamma^*} |\tau(\omega+\gamma)|^2 \leq 1 \quad \textit{for all } \omega \in \mathbb{T}^n,$$

(B) There exist rational trigonometric polynomials q_ℓ such that for all $\gamma \in \Gamma^*$ and $\omega \in \mathbb{T}^n$:

$$\tau(\omega)\overline{\tau(\omega+\gamma)} + \sum_{\ell=1}^{r} q_\ell(\omega)\overline{q_\ell(\omega+\gamma)} = \begin{cases} 1 & \textit{if } \gamma = 0 \\ 0 & \textit{otherwise.} \end{cases} \tag{5.10}$$

Moreover, provided that either (A) or (B) holds, there exist a potentially different set of rational trigonometric polynomials $q_\ell, 1 \leq \ell \leq r$ such that for all $\gamma \in \Gamma^*$ and $\omega \in \mathbb{T}^n$, Equation (5.10) holds, with $r \leq 2^n(1+\mathcal{Q})$. When one of (A) or (B) holds, the wavelet system defined by the combined MRA mask $(\tau, q_1, \ldots, q_r)$ is a tight wavelet frame.

This corollary is quite similar to the result [11, Thm. 2.2], but both statements here are weaker than the ones that appear in that theorem. In particular, the analogous statement for (A) in [11] requires the existence of an sos representation for $1 - \sum_\gamma |\tau^\gamma|^2$, but their result guarantees the existence of q_ℓ which are trigonometric polynomials in (B).

We apply this corollary to construct a tight wavelet frame using the lowpass mask considered in Example 4.2, for which $f(\tau;\cdot)$ has no sos representation.

Example 5.1. We extend Example 4.2 to construct a tight wavelet frame. Since $f(\tau;\cdot)$ has an sors representation, as we saw in Example 4.2, by Result 1.9(b), it has such a representation with at most $2^3 = 8$ generators. By Lemma 1.2(b) and Remark 5.1, f has a $\mathcal{G}$-invariant sors representation with at most 8 generators. Let $g_1, \ldots, g_8$ be these rational trigonometric polynomials. Then the wavelet system defined by the combined MRA mask $(\tau, q_{1,1}, \ldots, q_{1,8}, q_{2,(0,0,0)}, \ldots, q_{2,(1,1,1)})$ is a tight wavelet frame, with highpass masks

$$\begin{aligned} q_{1,j}(\omega) &= \tau(\omega)\overline{g_j(2\omega)} && 1 \le j \le 8, \\ q_{2,\nu}(\omega) &= 2^{-3/2}e^{i\omega\cdot\nu} - \tau(\omega)\overline{\tau_\nu(2\omega)} && \nu \in \{0,1\}^3, \end{aligned}$$

where τ_ν are the polyphase components of τ, as usual.

The following proposition tells us about the vanishing moments of highpass masks constructed using this approach. This is analogous to Theorem 1.1 for the UEP-based construction.

Proposition 5.1 (VMs of OEP Highpass Masks)**.** *In Theorem 5.1, let τ have accuracy number $a > 0$, $f(S, \tau; \cdot)$ have vanishing moments m, and $S - S(\mathcal{M}^T\cdot)|\tau|^2$ have vanishing moments ℓ. If (A) or (B) holds in that theorem, then the highpass masks $q_j, 1 \le j \le r$ in (B) have at least $\lfloor \ell/2 \rfloor \ge \lfloor \min\{a, m/2\} \rfloor$ vanishing moments.*

Proof. Rearranging the OEP conditions (5.2) with $\gamma = 0$, we get

$$S(\omega) - S(\mathcal{M}^T\omega)|\tau(\omega)|^2 = \sum_{j=1}^{r} |q_j(\omega)|^2,$$

so if the left-hand side is $O(|\omega|)^\ell$ for $\omega \approx 0$, then $q_j = O(|\omega|^{\ell/2})$ there, for all $1 \le j \le r$. If $f(S, \tau; \omega) = O(|\omega|^m)$ for $\omega \approx 0$, then

$$\frac{1}{S(\mathcal{M}^T\omega)} - \frac{|\tau(\omega)|^2}{S(\omega)} = O(|\omega|^m) + \sum_{\gamma \in \Gamma^* \setminus \{0\}} \frac{|\tau(\omega+\gamma)|^2}{S(\omega+\gamma)},$$

which is $O(|\omega|^{\min\{m,2a\}})$ for $\omega \approx 0$. Then

$$\begin{aligned} S(\omega) - S(\mathcal{M}^T\omega)|\tau(\omega)|^2 &= S(\omega)S(\mathcal{M}^T\omega)\left(\frac{1}{S(\mathcal{M}^T\omega)} - \frac{|\tau(\omega)|^2}{S(\omega)}\right) \\ &= S(\omega)S(\mathcal{M}^T\omega)O(|\omega|^{\min\{m,2a\}}) \text{ for } \omega \approx 0, \end{aligned}$$

which means that $\ell \geq \min\{m, 2a\}$. This completes the proof. □

5.1.2 Extending the Setting

If we move away from the rational trigonometric polynomial setting in Theorems 5.1 and 5.2, we observe that the argument that the OEP conditions imply the oblique sub-QMF condition (i.e., (B) implies (A)) is still valid, provided that we restrict to a set where the functions are all defined. We state some generalizations of this implication in the proposition below, and its proof makes clear the relationship between the set on which Equation (5.1) is guaranteed to hold, given the sets on which S is nonzero and Equation (5.2) holds.

Proposition 5.2. *Let S be a 2π-periodic function which is nonnegative on $\mathbb{T}^n$, and let $\mathcal{N} = \{\omega \in \mathbb{T}^n : S(\omega) \neq 0\}$. Let $\tau, q_\ell, 1 \leq \ell \leq r$ be 2π-periodic functions, and let $\mathcal{O} \subseteq \mathbb{T}^n$ be the set such that Equation (5.2) holds for all $\gamma \in \Gamma^*$ and for all $\omega \in \mathcal{O}$. Then the oblique sub-QMF condition, the inequality in (5.1), holds for all $\omega \in \mathcal{S}$, where $\mathcal{S}$ is a subset of $\mathbb{T}^n$ with the following properties:*

(a) If $\mathcal{N}, \mathcal{O}$ are open and dense, then so is $\mathcal{S}$.

(b) If $\mathcal{N}^c, \mathcal{O}^c$ have measure zero, then so does $\mathcal{S}^c$.

(c) If $\omega \in \mathcal{S}$, and $\omega' \in \mathbb{T}^n$ is such that $\omega' \equiv \omega + \gamma \pmod{2\pi\mathbb{Z}^n}$ for some $\gamma \in \Gamma^*$, then $\omega' \in \mathcal{S}$.

Proof. For a 2π-periodic function f, let $\mathcal{N}(f) := \{\omega \in \mathbb{T}^n : f(\omega) \neq 0\}$, and let $\mathcal{N}^* := \mathcal{N}(S(\mathcal{M}^T\cdot)) \cap \bigcap_{\gamma\in\Gamma^*} \mathcal{N}(S^\gamma)$. We will prove that the assumptions on $\mathcal{N}$ in (a) and (b) above imply the same for $\mathcal{N}^*$.

(a) $\mathcal{M}$ is a dilation matrix, so it is invertible, as is $\mathcal{M}^T$. This means that

$$\begin{aligned}\mathcal{N}(S(\mathcal{M}^T\cdot)) &= \{\omega \in \mathbb{T}^n : S(\mathcal{M}^T\omega) \neq 0\}\\ &= \{\mathcal{M}^{-T}\omega \in \mathbb{T}^n : \omega \equiv \omega' \,(\mathrm{mod}\, 2\pi\mathbb{Z}^n), \omega' \in \mathcal{N}\}\\ &= \mathcal{M}^{-T}\left(\bigcup_{k\in\mathbb{Z}^n}(\mathcal{N}+2\pi k)\right)\cap\mathbb{T}^n =: A\cap\mathbb{T}^n.\end{aligned}$$

Then if $\mathcal{N}$ is open and dense in $\mathbb{T}^n$, $A' := \bigcup_{k\in\mathbb{Z}^n}(\mathcal{N}+2\pi k)$ is open and dense in $\mathbb{R}^n$. By continuity of $\mathcal{M}^T$, $\mathcal{M}^{-T}$ is an open map, so $A = \mathcal{M}^{-T}(A')$ is open, and $\mathcal{N}(S(\mathcal{M}^T\cdot))$ is relatively open in $\mathbb{T}^n$. The set A must also be dense in $\mathbb{R}^n$, since otherwise there is an open ball $U \subseteq \mathbb{R}^n$ such that $A\cap U = \varnothing$, in which case $A'\cap\mathcal{M}^T(U) = \varnothing$. By continuity of $\mathcal{M}^{-T}$, $\mathcal{M}^T$ is an open map, so $\mathcal{M}^T(U)$ is open, which contradicts the density of A'. This proves that $\mathcal{N}(S(\mathcal{M}^T\cdot))$ is open and dense. Since the translations $\omega \mapsto \omega+\gamma$ are all continuous and continuously invertible, similar arguments show that $\mathcal{N}(S^\gamma)$ is open and dense. Since the intersection of finitely many open dense sets is again open and dense, $\mathcal{N}^*$ satisfies these properties when $\mathcal{N}$ does.

(b) Using the calculation in (a), properties of pullbacks, and De Morgan's Law,

$$\mathcal{N}(S(\mathcal{M}^T\cdot))^c\cap\mathbb{T}^n = \mathcal{M}^{-T}\left(\bigcap_{k\in\mathbb{Z}^n}(\mathcal{N}+2\pi k)^c\right)\cap\mathbb{T}^n \subseteq \mathcal{M}^{-T}(\mathcal{N}^c)\cap\mathbb{T}^n.$$

For any $A\subseteq\mathbb{R}^n$, $\mathrm{meas}(\mathcal{M}^{-T}A) = \mathcal{Q}^{-1}\mathrm{meas}(A)$, so $\mathrm{meas}(\mathcal{N}^c) = 0$ clearly implies that $\mathrm{meas}(\mathcal{M}^{-T}(\mathcal{N}^c)\cap\mathbb{T}^n) = 0$, and therefore $\mathrm{meas}(\mathcal{N}(S(\mathcal{M}^T\cdot))^c\cap\mathbb{T}^n) = 0$. The argument is similar for $\mathcal{N}(S^\gamma)$, and since $\mathcal{N}^*$ is the intersection of these sets,

$$\mathrm{meas}((\mathcal{N}^*)^c) \le \mathrm{meas}(\mathcal{N}(S(\mathcal{M}^T\cdot))^c) + \sum_{\gamma\in\Gamma^*}\mathrm{meas}(\mathcal{N}(S^\gamma)^c) = 0.$$

If Equation (5.2) holds on $\mathcal{O}$, then by part (ii) of the proof of Theorem 5.1, we see that the inequality of (5.1) holds for all $\omega\in\mathcal{S} = \mathcal{N}^*\cap\mathcal{O}$. Thus, the various assumptions on $\mathcal{N}$ and $\mathcal{O}$ lead to the same properties for $\mathcal{S}$, and in particular, $\mathcal{S}$ is nonempty under either of the sets of assumptions in (a) or (b).

The last property follows since both sides of Equation (5.1) are $\mathcal{G}$-invariant. $\square$

Inspecting the argument for Theorem 5.2, we see that this does not explicitly require that S is a rational trigonometric polynomial, besides ensuring the existence of the q_ℓ with this property from Theorem 5.1. However, the argument by induction is complicated in this setting, because the sets on which inequalities (5.8) and (5.9) hold may no longer be all of $\mathbb{R}^n$. After addressing these technical considerations, we have the following proposition.

Proposition 5.3. *Let τ be a trigonometric polynomial lowpass mask, let $q_\ell, 1 \leq \ell \leq r$ be rational trigonometric polynomial masks, and let $S \in L^\infty(\mathbb{T}^n)$ be a 2π-periodic, nonnegative function which is continuous at 0, with $S(0) = 1$. If $1/S$ also belongs to $L^\infty(\mathbb{T}^n)$, and Equation (5.2) holds for all $\gamma \in \Gamma^*$ and $\omega \in \mathbb{T}^n$, then the wavelet system defined by the combined MRA mask $(\tau, q_1, \ldots, q_r)$ is a tight wavelet frame.*

We first state a technical lemma which is used in the proof of this proposition.

Lemma 5.2. *Suppose τ is a trigonometric polynomial lowpass mask, and let S be a 2π-periodic, nonnegative, measurable function, such that $\mathrm{meas}(\mathcal{S}^c) = 0$, where $\mathcal{S}$ is the set on which Equation (5.1) holds. Let f_j be defined as in Equation (5.6) for all $j \geq 0$. Then there exists $\mathcal{D} \subseteq \mathbb{T}^n$ with $\mathrm{meas}(\mathcal{D}^c) = 0$ such that for all $j \geq 0$, $[f_j, f_j](\omega) \leq 1/S(\omega)$ for all $\omega \in \mathcal{D}$.*

This lemma will be proved shortly. Now we give the proof of the proposition.

Proof of Proposition 5.3: As in the proof of Theorem 5.2, we want to apply Result 1.7. We prove (i), (ii), (a), and (b) as we did in the proof of Theorem 5.2, since these proofs do not make use of the assumption that S is a rational trigonometric polynomial. Thus, the main change is in the proof of (c). Since $1/S \in L^\infty(\mathbb{T}^n)$, we see that $\mathcal{N}^c = \{\omega \in \mathbb{T}^n : S(\omega) = 0\}$ has measure zero. Then by Proposition 5.2(b), using $\mathcal{O} = \mathbb{T}^n$, we see that Equation (5.1) holds for all ω in some set $\mathcal{S} \subseteq \mathbb{T}^n$ with $\mathrm{meas}(\mathcal{S}^c) = 0$. Define f_j as in Equation (5.6) for all $j \geq 0$. Applying Lemma 5.2, we see that there is some set $\mathcal{D} \subseteq \mathbb{T}^n$ such that $[f_j, f_j](\omega) \leq 1/S(\omega)$ for all $\omega \in \mathcal{D}$, where $\mathrm{meas}(\mathcal{D}^c) = 0$.

Applying Fatou's Lemma as in the proof of Theorem 5.2, we see that $[\hat{\phi}, \hat{\phi}](\omega) \leq$

$\liminf_{j\to\infty}[f_j, f_j](\omega)$ for all $\omega \in \mathbb{R}^n$, so

$$[\hat{\phi}, \hat{\phi}](\omega) \leq 1/S(\omega) \text{ for all } \omega \in \mathcal{D}. \tag{5.11}$$

In particular, this bound holds almost everywhere, so $[\hat{\phi}, \hat{\phi}] \in L^\infty(\mathbb{T}^n)$ from the assumption on $1/S$. $\square$

To finish the proof of this proposition, we give the proof of Lemma 5.2. Since the proof is technical, we begin with a sketch of the argument. We would like to argue by induction as in Theorem 5.2, but see that this will be complicated by Equations (5.8) and (5.9), which no longer hold on all of $\mathbb{T}^n$. If the desired set $\mathcal{D}$ exists, then Equation (5.8) will hold whenever $\mathcal{M}^{-T}\omega + \gamma \in \mathcal{D}$, and by definition, Equation (5.9) will hold whenever $\mathcal{M}^{-T}\omega \in \mathcal{S}$, so we seek a set $\mathcal{D}$ such that for $\omega \in \mathcal{D}$, these two properties will hold. The second condition suggests $\mathcal{D} = \mathcal{M}^T\mathcal{S}$, but then if $\mathcal{M}^{-T}\omega + \gamma \in \mathcal{M}^T\mathcal{S}$, we see that $\omega \in (\mathcal{M}^T)^2(\mathcal{S} - \mathcal{M}^{-T}\gamma)$ (once these sets are appropriately defined), so we might take $\mathcal{D}$ to be the intersection of these. Continuing with this idea gives the following set:

$$\mathcal{D} = (\mathcal{M}^T\mathcal{S}) \cap \bigcap_{k=1}^{\infty} (\mathcal{M}^T)^{k+1} \left[\bigcap_{\gamma_1,\ldots,\gamma_k \in \Gamma^*} \left(\mathcal{S} - \sum_{j=1}^{k} (\mathcal{M}^{-T})^j \gamma_j \right) \right]. \tag{5.12}$$

Now we make things precise.

Proof of Lemma 5.2: In Equation (5.12), we use the following definitions: For $\mathcal{C} \subseteq \mathbb{T}^n$, $c \in \mathbb{R}^n$, $A \in M_n(\mathbb{R})$, let $\mathcal{C} + c := \{\omega \in \mathbb{T}^n : \exists \omega' \in \mathcal{C} \text{ such that } \omega \equiv \omega' + c \,(\mathrm{mod}\, 2\pi\mathbb{Z}^n)\}$, $A\mathcal{C} := \{\omega \in \mathbb{T}^n : \exists \omega' \in \mathcal{C} \text{ such that } \omega \equiv A\omega' \,(\mathrm{mod}\, 2\pi\mathbb{Z}^n)\}$.

Since the quantities in the inequalities (5.8) and (5.9) are 2π-periodic, we want to show that $\mathcal{D}$ above has the following properties:

1. Whenever $\omega \in \mathcal{D}$ and $\gamma \in \Gamma^*$, there is some $\omega' \in \mathcal{D}$ such that $\mathcal{M}^{-T}\omega + \gamma \equiv \omega' \,(\mathrm{mod}\, 2\pi\mathbb{Z}^n)$

2. Whenever $\omega \in \mathcal{D}$, there is some $\omega' \in \mathcal{S}$ such that $\mathcal{M}^{-T}\omega \equiv \omega' \,(\mathrm{mod}\, 2\pi\mathbb{Z}^n)$.

Proof of Property 2: For $\omega \in \mathcal{D}$, $\omega \in \mathcal{M}^T\mathcal{S}$, so by definition, there is some $\omega' \in \mathcal{S}$, $\ell \in \mathbb{Z}^n$ such that $\omega = \mathcal{M}^T\omega' + 2\pi\ell$. Then $\mathcal{M}^{-T}\omega = \omega' + 2\pi\mathcal{M}^{-T}\ell \equiv \omega' + \gamma \,(\mathrm{mod}\, 2\pi\mathbb{Z}^n)$ for some $\gamma \in \Gamma^*$. Using Proposition 5.2(c) (or just by $\mathcal{G}$-invariance of both sides of Equation (5.1)),

$\mathcal{M}^{-T}\omega \equiv \omega'' \pmod{2\pi\mathbb{Z}^n}$ for some $\omega'' \in \mathcal{S}$. This proves property 2.

Proof of Property 1: Suppose for some $\gamma \in \Gamma^*$ that we can show that there is some $\omega_0 \in H_0 := \mathcal{M}^T\mathcal{S}$, and for all $k \geq 1$, $\omega_k \in H_k := (\mathcal{M}^T)^{k+1}\left[\bigcap_{\gamma_j \in \Gamma^*}\left(\mathcal{S} - \sum_{j=1}^{k}(\mathcal{M}^{-T})^j\gamma_j\right)\right]$, such that $\mathcal{M}^{-T}\omega + \gamma \equiv \omega_k \pmod{2\pi\mathbb{Z}^n}$ for all $k \geq 0$. Then if we let $\omega' \in \mathbb{T}^n$ be such that $\mathcal{M}^{-T}\omega + \gamma \equiv \omega' \pmod{2\pi\mathbb{Z}^n}$, we see that $\omega' \equiv \omega_k \pmod{2\pi\mathbb{Z}^n}$ for all $k \geq 0$, which implies that $\omega' \in H_k$ for all $k \geq 0$, using the definition of $\mathcal{AC}$ above. This means that $\omega' \in \mathcal{D}$ as desired. So to prove property 1, it is sufficient for us to show that for $\omega \in \mathcal{D}$, $\gamma \in \Gamma^*$, and each $k \geq 0$, there is some $\omega_k \in H_k$ such that $\mathcal{M}^{-T}\omega + \gamma \equiv \omega_k \pmod{2\pi\mathbb{Z}^n}$.

For $\omega \in \mathcal{D}$ and $k \geq 2$, $\omega \in H_k$, so there is an $\omega' \in H_k' := \bigcap_{\gamma_j \in \Gamma^*}\left(\mathcal{S} - \sum_{j=1}^{k}(\mathcal{M}^{-T})^j\gamma_j\right)$ such that $\omega = (\mathcal{M}^T)^{k+1}\omega' + 2\pi\ell$, $\ell \in \mathbb{Z}^n$. Then for any $\gamma \in \Gamma^*$, $\mathcal{M}^{-T}\omega + \gamma = (\mathcal{M}^T)^k\omega' + 2\pi\mathcal{M}^{-T}\ell + \gamma = (\mathcal{M}^T)^k\omega' + \gamma' + 2\pi\ell'$, for some $\gamma' \in \Gamma^*$ and $\ell' \in \mathbb{Z}^n$. Provided that for any $\gamma' \in \Gamma^*$, $\omega' + (\mathcal{M}^{-T})^k\gamma' \equiv \omega''$ for some $\omega'' \in H_{k-1}'$, this means that $\mathcal{M}^{-T}\omega + \gamma \equiv (\mathcal{M}^T)^k(\omega' + (\mathcal{M}^{-T})^k\gamma') \equiv (\mathcal{M}^T)^k\omega'' \equiv \omega_{k-1} \pmod{2\pi\mathbb{Z}^n}$, where $\omega_{k-1} \in H_{k-1}$.

Now we want to show that $\omega' \in H_k'$ means that for any $\gamma' \in \Gamma^*$, $\omega' + (\mathcal{M}^{-T})^k\gamma' \equiv \omega''$ for some $\omega'' \in H_{k-1}'$. By definition of H_k', for any $\gamma_1, \ldots, \gamma_k \in \Gamma^*$, there is some $\eta \in \mathcal{S}$ such that $\omega' \equiv \eta - \sum_{j=1}^{k}(\mathcal{M}^{-T})^j\gamma_j \pmod{2\pi\mathbb{Z}^n}$. But if $\gamma_k = \gamma'$, $\omega' + (\mathcal{M}^{-T})^k\gamma' \equiv \eta - \sum_{j=1}^{k-1}(\mathcal{M}^{-T})^j\gamma_j$. Since for all $\gamma_1, \ldots, \gamma_{k-1} \in \Gamma^*$, this congruence holds for some $\eta \in \mathcal{S}$, if we let $\omega'' \in \mathbb{T}^n$ be such that $\omega'' \equiv \omega' + (\mathcal{M}^{-T})^k\gamma' \pmod{2\pi\mathbb{Z}^n}$, $\omega'' \in H_{k-1}'$. This shows that $\mathcal{M}^{-T}\omega + \gamma \equiv \omega_{k-1} \pmod{2\pi\mathbb{Z}^n}$ for some $\omega_{k-1} \in H_{k-1}$, for all $k \geq 2$.

Since $\omega \in \mathcal{D}$ means that $\omega \in H_1$, there is an $\omega' \in H_1'$ such that $\omega = (\mathcal{M}^T)^2\omega' + 2\pi\ell$, $\ell \in \mathbb{Z}^n$. Then for any $\gamma \in \Gamma^*$, $\mathcal{M}^{-T}\omega + \gamma = \mathcal{M}^T\omega' + \gamma' + 2\pi\ell'$, for some $\gamma' \in \Gamma^*$ and $\ell' \in \mathbb{Z}^n$. But since $\omega' \in H_1'$, there is some $\eta \in \mathcal{S}$ such that $\omega' \equiv \eta - \mathcal{M}^{-T}\gamma' \pmod{2\pi\mathbb{Z}^n}$, in which case $\omega' + \mathcal{M}^{-T}\gamma' \equiv \eta \pmod{2\pi\mathbb{Z}^n}$. Thus $\mathcal{M}^{-T}\omega + \gamma \equiv \mathcal{M}^T(\omega' + \mathcal{M}^{-T}\gamma') \equiv \mathcal{M}^T\eta \pmod{2\pi\mathbb{Z}^n}$, so letting $\omega_0 \in \mathbb{T}^n$ such that $\omega_0 \equiv \mathcal{M}^{-T}\omega + \gamma$, $\omega_0 \in H_0 = \mathcal{M}^T\mathcal{S}$. This proves that for all $\gamma \in \Gamma'$, there exist $\omega_k \in H_k$ for all $k \geq 0$ such that $\mathcal{M}^{-T}\omega + \gamma \equiv \omega_k \pmod{2\pi\mathbb{Z}^n}$, which completes the proof of property 1.

Note that $\mathcal{D}$ is a countable intersection of sets, and since $\text{meas}(H_k^c) = 0$ for all $k \geq 0$ (the argument for this is very similar to the proof of Proposition 5.2(b)), $\text{meas}(\mathcal{D}^c) = 0$ also. Now, arguing by induction as in the proof of Theorem 5.2, we see that for $\omega \in \mathcal{D}$ and $\gamma \in \Gamma^*$, by property 1, $\mathcal{M}^{-T}\omega + \gamma \equiv \omega' \pmod{2\pi\mathbb{Z}^n}$ for some $\omega' \in \mathcal{D}$, so by 2π-periodicity and the induction hypothesis, Equation (5.8) holds. By property 2, 2π-periodicity, and the definition of $\mathcal{S}$, Equation (5.9) holds. Then the argument of the induction is correct for ω restricted to the set $\mathcal{D}$, which completes the proof. □

5.2 Scaling Oblique Laplacian Pyramid Matrices

We now consider another perspective on Theorem 5.1, which is similar to the one in [37], but rather than working with the Laplacian pyramid (LP) matrix coming from the polyphase components as was done there, in keeping with the current work, we consider the LP matrix with $\mathcal{G}$-vector columns[1]. Let us start by describing the idea in [37], which considers constructions based on the UEP. Given a lowpass mask τ, let $H(\omega) = [\tau(\omega+\gamma)]_{\gamma\in\Gamma^*}$. Then we define the Laplacian pyramid matrix $\Phi_\tau(\omega) = [H(\omega)\ (I - H(\omega)H(\omega)^*)X(\omega)]$, which is a $\mathcal{Q}\times(\mathcal{Q}+1)$ matrix with trigonometric polynomial entries and $\mathcal{G}$-vector columns, and $X(\omega) = [\mathcal{Q}^{-1/2}e^{i(\omega+\gamma)\cdot\nu}]_{\gamma\in\Gamma^*,\nu\in\Gamma}$ is the Fourier transform matrix. When $\Phi_\tau(\omega)\Phi_\tau(\omega)^* = I$, then by inspecting the entries of this matrix product, we see that τ satisfies the UEP conditions with the highpass masks $q_\nu(\omega) = \mathcal{Q}^{-1/2}e^{i\omega\cdot\nu} - \tau(\omega)\overline{\tau_\nu(\mathcal{M}^T\omega)}$. However, this requires τ to satisfy the restrictive QMF condition. Even when this condition does not hold, however, we can see that

$$\Phi_\tau(\omega)\begin{bmatrix} H(\omega)^* \\ X(\omega)^* \end{bmatrix} = [H(\omega)\ (I - H(\omega)H(\omega)^*)X(\omega)]\begin{bmatrix} H(\omega)^* \\ X(\omega)^* \end{bmatrix} = I,$$

so Φ_τ has a right-inverse. But the latter matrix does not have the structure of a wavelet filter bank: besides the lowpass mask in the first row, denoting the trigonometric polynomials generating the rows of this matrix after the first by $\tilde{q}_\nu$, for $\nu\in\Gamma$, we have $\tilde{q}_\nu(\omega) = \mathcal{Q}^{-1/2}e^{-i\omega\cdot\nu}$, so $\tilde{q}_\nu(0) = \mathcal{Q}^{-1/2} \neq 0$, and therefore $\tilde{q}_\nu$ are not wavelet masks. To correct this, we try scaling the matrix Φ_τ to get a new filter bank satisfying the UEP conditions. If we can find a matrix $D(\omega)$ with trigonometric polynomial entries such that $\Phi_\tau(\omega)D(\omega)\Phi_\tau(\omega)^* = I$, then supposing this has a factorization as $B(\mathcal{M}^T\omega)B(\mathcal{M}^T\omega)^*$ for some $(\mathcal{Q}+1)\times(r+1)$ matrix

[1] To translate between these, consider the Fourier transform matrix $X(\omega) = [\mathcal{Q}^{-1/2}e^{i(\omega+\gamma)\cdot\nu}]_{\gamma\in\Gamma^*,\nu\in\Gamma}$. For a mask $g(\omega)$, if $G(\omega) = [g(\omega+\gamma)]_{\gamma\in\Gamma^*}$, then $[g_\nu(\mathcal{M}^T\omega)]_{\nu\in\Gamma} = X(\omega)^*G(\omega)$ (c.f. Equation (1.6)).

$B(\omega)$ with trigonometric polynomial entries, the product $\Phi'(\omega) = \Phi_\tau(\omega)B(\mathcal{M}^T\omega)$ will have $\mathcal{G}$-vector columns and satisfy $\Phi'(\omega)\Phi'(\omega)^* = I$. Provided Φ' still has a lowpass mask generating its first column and highpass masks for the remaining columns, we will have a new collection of masks satisfying the UEP conditions, and an associated tight wavelet frame generated by these. Now, we translate these ideas to the case of the OEP.

Suppose that S is a rational trigonometric polynomial satisfying $S(0) = 1$, and let τ be a lowpass mask satisfying the oblique sub-QMF condition with S. Recalling that $\Sigma(\omega) = \operatorname{diag}(S(\omega+\gamma))_{\gamma\in\Gamma^*}$, we define the oblique Laplacian pyramid (OLP) matrix $\Phi_{S,\tau}(\omega) = [H(\omega)\ (\Sigma(\omega) - S(\mathcal{M}^T\omega)H(\omega)H(\omega)^*)X(\omega)]$, where $X(\omega)$ is the Fourier transform matrix as above. Then

$$\Phi_{S,\tau}(\omega)\begin{bmatrix} S(\mathcal{M}^T\omega)H(\omega)^* \\ X(\omega)^* \end{bmatrix} = \Sigma(\omega),$$

so inverting $\Sigma(\omega)$, the matrix $\Phi_{S,\tau}$ has a right-inverse almost everywhere, in particular, wherever $S(\omega+\gamma) \neq 0$ for all $\gamma \in \Gamma^*$. However, the second matrix is once again not a wavelet filter bank, so as before, we will attempt to correct this by scaling the masks in $\Phi_{S,\tau}$ in order to get a new collection of masks satisfying the OEP conditions.

Let $a(\mathcal{M}^T\omega) = S(\mathcal{M}^T\omega)(2 - S(\mathcal{M}^T\omega)H(\omega)^*\Sigma(\omega)^{-1}H(\omega))$. Then

$$\begin{aligned}
&\Phi_{S,\tau}(\omega)\begin{bmatrix} a(\mathcal{M}^T\omega) & 0 \\ 0 & X(\omega)^*\Sigma(\omega)^{-1}X(\omega) \end{bmatrix}\Phi_{S,\tau}(\omega)^* \\
&\quad = a(\mathcal{M}^T\omega)H(\omega)H(\omega)^* + \Sigma(\omega) - 2S(\mathcal{M}^T\omega)H(\omega)H(\omega)^* \\
&\qquad + S(\mathcal{M}^T\omega)^2(H(\omega)^*\Sigma(\omega)^{-1}H(\omega))H(\omega)H(\omega)^* \\
&\quad = \Sigma(\omega) + [a(\mathcal{M}^T\omega) + S(\mathcal{M}^T\omega)(-2 + S(\mathcal{M}^T\omega)H(\omega)^*\Sigma(\omega)^{-1}H(\omega))]H(\omega)H(\omega)^* \\
&\quad = \Sigma(\omega).
\end{aligned}$$

Note that $\Sigma(\omega)^{-1}$ is always well-defined as a rational trigonometric polynomial matrix so long as $S \not\equiv 0$, but if S has zeroes, then $\Sigma(\omega)^{-1}$ will have poles. The assumptions of Theorem 5.2 on S also preclude this behavior, when they hold.

Then to obtain masks satisfying the OEP conditions, we want a factorization of the scaling matrix as

$$\begin{bmatrix} a(\mathcal{M}^T\omega) & 0 \\ 0 & X(\omega)^*\Sigma(\omega)^{-1}X(\omega) \end{bmatrix} = B(\mathcal{M}^T\omega) \begin{bmatrix} S(\mathcal{M}^T\omega) & 0 \\ 0 & I \end{bmatrix} B(\mathcal{M}^T\omega)^*, \tag{5.13}$$

in which case $\Phi'(\omega) = \Phi_{S,\tau}(\omega)B(\mathcal{M}^T\omega)$ will have $\mathcal{G}$-vector columns, and provided its first column is generated by a lowpass mask and the rest are highpass, we will have a new collection of masks satisfying the OEP conditions, since

$$\Phi'(\omega) \begin{bmatrix} S(\mathcal{M}^T\omega) & 0 \\ 0 & I \end{bmatrix} \Phi'(\omega)^* = \Phi_{S,\tau}(\omega) \begin{bmatrix} a(\mathcal{M}^T\omega) & 0 \\ 0 & X(\omega)^*\Sigma(\omega)^{-1}X(\omega) \end{bmatrix} \Phi_{S,\tau}(\omega)^* = \Sigma(\omega).$$

These are just the matrix version of the OEP conditions, as discussed in Remark 5.2.

Now we see that different factorizations of the scaling matrix lead to different tight wavelet frame constructions. In the proof of Theorem 5.1, we do not modify the lowpass mask, and add highpass masks corresponding to the sors generators for $a(\mathcal{M}^T\omega) - S(\mathcal{M}^T\omega) = S(\mathcal{M}^T\omega)^2(1/S(\mathcal{M}^T\omega) - H(\omega)^*\Sigma(\omega)^{-1}H(\omega))$. Then we might write the factorization of Equation (5.13) with

$$B(\mathcal{M}^T\omega) = \begin{bmatrix} 1 & S(\mathcal{M}^T\omega)G(\mathcal{M}^T\omega)^* & 0 \\ 0 & 0 & X(\omega)^*A(\omega) \end{bmatrix},$$

and $G(\omega)$, $A(\omega)$ are as in the proof of the theorem. Since the columns of $A(\omega)$ are $\mathcal{G}$-vectors, $X(\omega)^*A(\omega) = [(a_m)_\nu(\mathcal{M}^T\omega)]_{\nu\in\Gamma,1\le m\le M}$, where a_m is the rational trigonometric polynomial generating the mth column of $A(\omega)$.

If instead there is a square root for $a(\omega)/S(\omega)$, so that $a(\omega)/S(\omega) = |g(\omega)|^2$ for all $\omega \in \mathbb{T}^n$, then we might write the factorization of Equation (5.13) with

$$B(\mathcal{M}^T\omega) = \begin{bmatrix} g(\mathcal{M}^T\omega) & 0 \\ 0 & X(\omega)^*A(\omega) \end{bmatrix}.$$

This corresponds to modifying the lowpass mask to obtain $g(\mathcal{M}^T\cdot)\tau$.

A third possibility combines these two ideas, requiring a representation for $a(\omega)$ as $S(\omega)|g_0(\omega)|^2 + S(\omega)^2 \sum_{j=1}^{J} |g_j(\omega)|^2$, where $g_0(0) = 1$. Then we might write the factorization of Equation (5.13) with

$$B(\mathcal{M}^T\omega) = \begin{bmatrix} g_0(\mathcal{M}^T\omega) & S(\mathcal{M}^T\omega)G(\mathcal{M}^T\omega)^* & 0 \\ 0 & 0 & X(\omega)^*A(\omega) \end{bmatrix},$$

obtaining a modified lowpass mask $g_0(\mathcal{M}^T\cdot)\tau$, as well as new highpass masks corresponding to the generators $g_1, \ldots, g_J$.

As such, depending on the kinds of sors representations available for $a(\omega)$, and the criteria of the tight wavelet filter bank designer (such as whether or not the lowpass mask should be modified), some of these constructions may be preferable to others. Further investigation of possible factorizations of this scaling matrix may also lead to entirely new constructions.

5.3 Examples

In this section, we consider the case of lowpass masks associated with box spline refinable functions, and apply Theorems 5.1 and 5.2. The first is a simple example which has been well-studied, for example, in Example 2.1, and [42, Example 5.2].

Example 5.2 (Piecewise Linear Box Spline, $n = 2$)**.** Let $\tau(\omega) = 8^{-1}(1+e^{i\omega_1})(1+e^{i\omega_2})(1+e^{i(\omega_1+\omega_2)})$ be the lowpass mask associated with the piecewise linear box spline refinable function in dimension 2 with dyadic dilation. Note that τ has accuracy and flatness numbers 2. Suppose $S(\omega) = 1/B(\omega)$, where $B(\omega) = a + b(\cos(\omega_1) + \cos(\omega_2)) + c\cos(\omega_1 + \omega_2)$. We will see that there are ranges of values for a, b, c for which Theorem 5.1 applies, which demonstrates the flexibility afforded by the theorem above compared to that of [42], for which there is a single choice of S given. Note that so long as $a + 2b + c = 1$, the flatness number of S equals that of B, which is 2 whenever $b, c \neq 0$.

We compute

$$\frac{1}{S(2\omega)} - \sum_{\gamma\in\{0,\pi\}^n} \frac{|\tau(\omega+\gamma)|^2}{S(\omega+\gamma)} = \frac{3a-6b-3c}{8} - \left(\frac{a-4b+c}{8}\right)(\cos(2\omega_1)+\cos(2\omega_2)) - \left(\frac{a+2b-5c}{8}\right)\cos(2(\omega_1+\omega_2)). \tag{5.14}$$

This has an sos representation with two sos generators, making it equal to

$$\left|\frac{\sqrt{3a-12b+3c}}{4}(\sqrt{2/3}-\sqrt{1/6}(e^{-2i\omega_1}+e^{2i\omega_2}))\right|^2 + \left|\frac{\sqrt{3a-9c}}{4\sqrt{2}}(1-e^{2i(\omega_1+\omega_2)})\right|^2,$$

which is valid whenever $a \geq \max\{4b-c, 3c\}$. This ensures that $3a-12b+3c \geq 0$ and $3a-9c \geq 0$. In fact, since we assume that $S(0) = 1$, which gives $a = 1-2b-c$, these conditions become $6b \leq 1$, and $2b+4c \leq 1$, which are together sufficient to guarantee that $1/S(\omega)$ has an sos representation as

$$|\sqrt{1/3-3b/2}(\sqrt{2/3}-\sqrt{1/6}(e^{-i\omega_1}+e^{i\omega_2}))|^2 + \left|\frac{\sqrt{1/3-b/2-c}}{\sqrt{2}}(1-e^{i(\omega_1+\omega_2)})\right|^2 + \left|\frac{1}{3}(1+e^{i\omega_1}+e^{-i\omega_2})\right|^2.$$

That is, $1/3-3b/2 \geq 0$ and $1/3-b/2-c \geq 0$ whenever $6b \leq 1$ and $2b+4c \leq 1$.

But now we observe that when these are nonzero, both of the sos generators for $f(S,\tau;\omega)$ in Equation (5.14) have exactly 1 vanishing moment. In [42], a tight wavelet frame for this same lowpass mask is constructed with 2 vanishing moments for all wavelet masks, but there is only one possible choice of S considered there (with $a = 1/2, b = c = 1/6$). If we choose these same values of a, b, and c, we see that Equation (5.14) has right hand side equal to 0, and changing the sos representation for $1/S(\omega)$ using these values leads to the same tight wavelet frame as in [42]. In Example 2.1, this example was studied using the UEP, giving 7 wavelet masks, three of which have 1 vanishing moment, and the other four having 2 vanishing moments, and this construction was extended to any dimension. We extend this example to any dimension in Example 5.4 below. □

While a method for finding a rational trigonometric polynomial S satisfying the oblique

QMF condition was described in [42], which results in a construction with maximum vanishing moments, this requires computing the Fourier series of the autocorrelation function for ϕ, which is complicated. We show a relatively simple construction for S which assumes that S is of the form $[\prod_k s_k(\omega \cdot \xi_k)]^{-1}$, for a number of univariate trigonometric polynomials s_k equal to the number of distinct directions in the box spline. This gives an intermediate number of vanishing moments between 1 and the maximum, and equals the maximum when the box spline comes from the tensor product of univariate lowpass masks.

Example 5.3 (General Construction for Box Splines)**.** Let us consider $\mathcal{M} = 2I$, where I is the $n \times n$ identity matrix for spatial dimension n, and let Ξ be an $n \times m$ integer matrix. Let $\tau(\omega) = \prod_{k=1}^{d}(2^{-1}(1 + e^{i\omega\cdot\xi_k}))^{m_k}$, where $\{\xi_1, \dots, \xi_d\}$ are the distinct columns of Ξ, so that $m = \sum_{j=1}^{d} m_j$, where ξ_k is repeated m_k times for $1 \leq k \leq d$. We suppose that $\Xi = [\Xi_0, \Xi_1]$, where Ξ_0 is square and invertible mod 2. In particular, this means that $d \geq n$. Let Ξ_0^{-1} be a square integer matrix such that $\Xi_0^{-1}\Xi_0 \equiv I \pmod{2\mathbb{Z}^{n\times n}}$, and with slight abuse of notation, let the columns of Ξ_0^{-T} be $\xi_k^{-1}, 1 \leq k \leq n$.

Let $\ell \geq \min\{m_k : 1 \leq k \leq n\}$. Now we suppose that there exists a collection of univariate trigonometric polynomials $s_k, 1 \leq k \leq d$, such that

$$\begin{aligned} & s_k(\omega) > 0 \,\forall \omega \in \mathbb{T}, \text{ for all } 1 \leq k \leq d, \\ & \left.\begin{aligned} s_k(2\omega) - \textstyle\sum_{\gamma\in\{0,\pi\}} \cos^{2m_k}((\omega+\gamma)/2) s_k(\omega+\gamma) \geq 0 \,\forall\omega \in \mathbb{T} \\ s_k(2\omega) - \textstyle\sum_{\gamma\in\{0,\pi\}} \cos^{2m_k}((\omega+\gamma)/2) s_k(\omega+\gamma) = O(|\omega|^{2\ell}) \text{ for } \omega \approx 0 \end{aligned}\right\} \text{for } 1 \leq k \leq n, \text{ and} \\ & \left.\begin{aligned} s_k(2\omega) - \cos^{2m_k}(\omega/2) s_k(\omega) \geq 0 \,\forall\omega \in \mathbb{T} \\ s_k(2\omega) - \cos^{2m_k}(\omega/2) s_k(\omega) = O(|\omega|^{2\ell}) \text{ for } \omega \approx 0 \end{aligned}\right\} \text{for } n+1 \leq k \leq d. \end{aligned} \tag{5.15}$$

We will describe a method for obtaining trigonometric polynomials satisfying these properties in Remark 5.4. Let $S(\omega) = \left(\prod_{k=1}^{d} s_k(\omega \cdot \xi_k)\right)^{-1}$. We will show that for this S and τ, $f(S, \tau; \cdot)$ has $2\mu = 2\min\{m_k : 1 \leq k \leq n\}$ vanishing moments. Then since τ has accuracy number $a \geq \mu$ (and equal to this number when τ comes from the tensor product), by Proposition 5.1, the highpass masks satisfying the OEP conditions with this τ and S will have at least μ vanishing moments.

For $\omega \in \mathbb{T}^n, \gamma = [\gamma_i]_{i=1}^n \in \{0,\pi\}^n$, and $1 \le i \le d$, let $t_i(\omega,\gamma) = \cos^{2m_i}((\omega + \Xi_0^{-T}\gamma)\cdot \xi_i/2)s_i((\omega+\Xi_0^{-T}\gamma)\cdot\xi_i)$. Note that for $1 \le i \le n$, using $\Xi_0^{-1}\Xi_0 \equiv I \,(\mathrm{mod}\, 2\mathbb{Z}^{n\times n})$, we have that $(\omega + \Xi_0^{-T}\gamma)\cdot\xi_i \equiv \omega\cdot\xi_i + \gamma_i \,(\mathrm{mod}\, 2\pi)$, so for $1 \le i \le n$, using 2π-periodicity of s_i and π-periodicity of $\cos^{2m_i}(\cdot)$, we may write $t_i(\omega\cdot\xi_i,\gamma_i) = \cos^{2m_i}((\omega\cdot\xi_i+\gamma_i)/2)s_i(\omega\cdot\xi_i+\gamma_i)$, since these t_i do not depend on the full vectors ω and γ. In the computation below, we only write t_i in this way when we want to emphasize this lack of dependence.

Changing our set of representatives for $(2\pi\mathcal{M}^{-T}\mathbb{Z}^n/2\pi\mathbb{Z}^n)$ from $\{0,\pi\}^n$ to $\Xi_0^{-T}\{0,\pi\}^n$, and using the $2\pi\mathbb{Z}^n$-periodicity of these functions, we have

$$
\begin{aligned}
\frac{1}{S(2\omega)} - \sum_{\gamma\in\{0,\pi\}^n}\frac{|\tau(\omega+\gamma)|^2}{S(\omega+\gamma)} &= \frac{1}{S(2\omega)} - \sum_{\gamma\in\{0,\pi\}^n}\frac{|\tau(\omega+\Xi_0^{-T}\gamma)|^2}{S(\omega+\Xi_0^{-T}\gamma)} \\
&= \prod_{k=1}^d s_k(2\omega\cdot\xi_k) - \sum_{\gamma\in\{0,\pi\}^n}\prod_{j=1}^d \cos^{2m_j}((\omega+\Xi_0^{-T}\gamma)\cdot\xi_j/2)s_j((\omega+\Xi_0^{-T}\gamma)\cdot\xi_j) \\
&= \prod_{k=1}^d s_k(2\omega\cdot\xi_k) - \sum_{\gamma\in\{0,\pi\}^n}\left(\prod_{i=1}^n t_i(\omega\cdot\xi_i,\gamma_i)\right)\left(\prod_{j=n+1}^d t_j(\omega,\gamma)\right) \\
&= \prod_{k=1}^d s_k(2\omega\cdot\xi_k) - \left(\prod_{i=1}^n \sum_{\gamma_i\in\{0,\pi\}} t_i(\omega\cdot\xi_i,\gamma_i)\right)\left(\prod_{k=n+1}^d s_k(2\omega\cdot\xi_k)\right) \qquad (5.16) \\
&+ \left(\sum_{\gamma\in\{0,\pi\}^n}\prod_{i=1}^n t_i(\omega,\gamma)\right)\left(\prod_{k=n+1}^d s_k(2\omega\cdot\xi_k)\right) - \sum_{\gamma\in\{0,\pi\}^n}\left(\prod_{i=1}^d t_i(\omega,\gamma)\right), \qquad (5.17)
\end{aligned}
$$

where the middle two quantities are equal, so we have added 0 to get the last equation. Expanding Line (5.16) as a telescoping series, we get

$$
\begin{aligned}
&\left(\prod_{j=2}^d s_j(2\omega\cdot\xi_j)\right)\left(s_1(2\omega\cdot\xi_1) - \sum_{\gamma_1\in\{0,\pi\}} t_1(\omega\cdot\xi_1,\gamma_1)\right) + \\
&\left(\prod_{j=3}^d s_j(2\omega\cdot\xi_j)\right)\left(\sum_{\gamma_1\in\{0,\pi\}} t_1(\omega\cdot\xi_1,\gamma_1)\right)\left(s_2(2\omega\cdot\xi_2) - \sum_{\gamma_2\in\{0,\pi\}} t_2(\omega\cdot\xi_2,\gamma_2)\right) + \cdots
\end{aligned}
$$

Or in a more compact form,

$$\sum_{k=1}^{n}\left(\prod_{j=k+1}^{d} s_j(2\omega\cdot\xi_j)\right)\left(\prod_{i=1}^{k-1}\sum_{\gamma_i\in\{0,\pi\}} t_i(\omega\cdot\xi_i,\gamma_i)\right)\left(s_k(2\omega\cdot\xi_k)-\sum_{\gamma_k\in\{0,\pi\}} t_k(\omega\cdot\xi_k,\gamma_k)\right). \tag{5.18}$$

For Line (5.17), since both terms are being summed over $\gamma\in\{0,\pi\}^n$, and $s_k(2\cdot)$ is $\mathcal{G}$-invariant, this equals

$$\sum_{\gamma\in\{0,\pi\}^n}\left[\left(\prod_{i=1}^{n} t_i(\omega,\gamma)\right)\left(\prod_{k=n+1}^{d} s_k(2\omega\cdot\xi_k)\right)-\prod_{i=1}^{d} t_i(\omega,\gamma)\right].$$

Now we telescope within the brackets, obtaining

$$\sum_{\gamma\in\{0,\pi\}^n}\left[\sum_{k=n+1}^{d}\left(\prod_{j=k+1}^{d} s_j(2\omega\cdot\xi_j)\right)\left(\prod_{i=1}^{k-1} t_i(\omega,\gamma)\right)(s_k(2\omega\cdot\xi_k)-t_k(\omega,\gamma))\right]. \tag{5.19}$$

By the nonnegativity assumptions on the s_k in Equation (5.15), all of the factors in every term of (5.18) and (5.19) are nonnegative. Each factor in every term of (5.18) and (5.19) is a nonnegative univariate trigonometric polynomial, so applying the Fejér-Riesz Lemma several times, we have an sos representation for $f(S,\tau;\cdot)$ with n terms from (5.18) and $2^n(d-n)$ terms from (5.19), for a total of $n+2^n(d-n)$ sos generators. In fact, the terms from (5.18) are products of nonnegative univariate trigonometric polynomials of the form $g(2\omega\cdot\xi_i)$ for some $1\le i\le d$, so applying the Fejér-Riesz Lemma to $g=|p|^2$, we get that $g(2\omega\cdot\xi)=|p(2\omega\cdot\xi)|^2$, which means these sos generators are $\mathcal{G}$-invariant. On the other hand, in (5.19), this does not hold, but since our application of the Fejér-Riesz Lemma gave us trigonometric polynomials g_k for which (5.19) equals (switching the order of summation) $\sum_{k=n+1}^{d}\sum_{\gamma\in\{0,\pi\}^n}|g_k^{\gamma}(\omega)|^2$, applying Lemma 1.2(a) yields a representation for (5.19) as

$$\sum_{k=n+1}^{d}\sum_{\nu\in\{0,1\}^n}|(g_k)_\nu(\mathcal{M}^T\omega)|^2,$$

which is a $\mathcal{G}$-invariant sos with $2^n(d-n)$ terms.

If we apply the method in the proof of Theorem 5.1, we get a highpass mask $q_{1,j}$ for every sos generator of $f(S,\tau;\cdot)$, plus an additional highpass mask $q_{2,m}$ for every column of

$A(\omega)$. We just showed that $f(S,\tau;\cdot)$ has a $\mathcal{G}$-invariant sos representation with $n+2^n(d-n)$ generators, and since $1/S$ is a product of positive univariate trigonometric polynomials, applying the Fejér-Riesz Lemma d times gives $1/S=|q|^2$ for a trigonometric polynomial q. Using the method of proof in Lemma 5.1 for writing $A(\omega)$, $1/S$ has $J=1$ sos generators, so we get a matrix $A(\omega)$ with $\mathcal{Q}=2^n$ columns. This leads to a tight wavelet frame with $n+2^n(d-n+1)$ wavelet masks, using the method of Theorem 5.1.

Moreover, using the conditions in Equation (5.15), we observe that in (5.18), the factors where subtraction is taking place are $O(|\omega|^{2\ell})$ for $\omega\approx 0$, and this is true for the $\gamma=0$ term in (5.19) as well. For $\gamma\neq 0$, $s_k(2(0)\cdot\xi_k)-t_k(0,\gamma)=s_k(2(0)\cdot\xi_k)-\cos^{2m_k}(\Xi_0^{-T}\gamma\cdot\xi_k/2)s_k(\Xi_0^{-T}\gamma\cdot\xi_k)$ can be equal to 1 or 0 depending on $\Xi_0^{-T}\gamma\cdot\xi_k$, so the vanishing moments have to come from the factors $t_i(\omega,\gamma), 1\leq i\leq k-1$. Since $k\geq n+1$, this includes all of the factors $\cos^{2m_i}((\omega\cdot\xi_i+\gamma_i)/2)$. Then for any $\gamma\in\{0,\pi\}^n\setminus\{0\}$, some $\gamma_i=\pi$, and the factor of $\cos^{2m_i}((\omega\cdot\xi_i+\pi)/2)=\sin^{2m_i}(\omega\cdot\xi_i/2)$ means this term must be $O(|\omega|^{2m_i})\leq O(|\omega|^{2\mu})$ for $\omega\approx 0$, where $\mu=\min\{m_k:1\leq k\leq n\}$. Since $f(S,\tau;\cdot)$ is the sum of (5.18) and (5.19), and $\ell\geq\mu$, we see that $f(S,\tau;\cdot)$ has at least 2μ vanishing moments. Applying Proposition 5.1, the highpass masks in the wavelet system constructed in Theorem 5.1 will all have at least μ vanishing moments, since τ has accuracy number at least μ.

In [12], a construction based on the UEP obtained the same number of highpass masks generating a tight wavelet frame with any box spline refinable function, but their construction always has some masks having only one vanishing moment, whereas ours typically gives more. The next examples use the procedure in this example to show how this plays out in a few particular cases. □

Remark 5.4 (Finding s_k as in Equation (5.15))**.** We now show how s_k with the properties of Equation (5.15) may be found. Let $f_m(\omega)=\cos^{2m}(\omega/2)$, and let $s_{a,m,\ell}(\omega)=\sum_{k=0}^{\ell-1}c_k\cos(k\omega)$. We want to use $s_{1,m,\ell}$ to satisfy the requirements for $1\leq k\leq n$ in Equation (5.15), and $s_{2,m,\ell}$ to satisfy the requirements for $n+1\leq k\leq d$ in this equation, where we will set $m=m_k$ in each case, and ℓ is a positive integer that we can choose to get more vanishing moments in the constructed highpass masks, up to $\mu=\min\{m_k:1\leq k\leq n\}$. In both cases, we focus on the the vanishing condition at $\omega=0$ (c.f. Equation (5.15)), since this yields algebraic (in fact, linear) conditions on the coefficients c_k, whereas the nonnegativity conditions are more complicated. Clearly the coefficients c_k depend on a, m, ℓ, but we suppress this from the notation. In many cases, we find that once we impose a sufficiently high degree of vanishing at $\omega = 0$, the nonnegativity assumptions in Equation (5.15) also hold, but this has not been proved in general.

Imposing the vanishing conditions of Equation (5.15) yields the equations below:

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}\omega}\right)^{2i}\left[s_{1,m,\ell}(2\omega) - f_m(\omega)s_{1,m,\ell}(\omega) - f_m(\omega+\pi)s_{1,m,\ell}(\omega+\pi)\right]_{\omega=0} = 0, \quad 0 \le i \le \ell - 1,\\ \left(\frac{\mathrm{d}}{\mathrm{d}\omega}\right)^{2i}\left[s_{2,m,\ell}(2\omega) - f_m(\omega)s_{2,m,\ell}(\omega)\right]_{\omega=0} = 0, \quad 0 \le i \le \ell - 1.\end{aligned} \tag{5.20}$$

Applying the general Leibniz formula for the derivatives of a product of two functions, we obtain a recurrence relation between $s_{1,m,\ell}^{(2i)}(0)$ and $\{\{f_m^{(2j)}(\gamma), s_{1,m,\ell}^{(2j)}(\gamma)\}_{j=0}^{i-1}\}_{\gamma\in\{0,\pi\}}$, as well as between $s_{2,m,\ell}^{(2i)}(0)$ and $\{f_m^{(2j)}(0), s_{2,m,\ell}^{(2j)}(0)\}_{j=0}^{i-1}$, using the fact that $f_m^{(2k+1)}(\gamma) = 0$ for $\gamma \in \{0, \pi\}$:

$$\begin{aligned}2^{2i}s_{1,m,\ell}^{(2i)}(0) - \sum_{k=0}^{i}\binom{2i}{2k}\left(f_m^{(2k)}(0)s_{1,m,\ell}^{(2(i-k))}(0) + f_m^{(2k)}(\pi)s_{1,m,\ell}^{(2(i-k))}(\pi)\right) = 0,\\ 2^{2i}s_{2,m,\ell}^{(2i)}(0) - \sum_{k=0}^{i}\binom{2i}{2k}f_m^{(2k)}(0)s_{2,m,\ell}^{(2(i-k))}(0) = 0.\end{aligned} \tag{5.21}$$

Expanding with complex exponentials using the binomial theorem, we find that

$$\begin{aligned}f_m^{(2j)}(\gamma) &= \left(\frac{\mathrm{d}}{\mathrm{d}\omega}\right)^{2j}\left(\frac{e^{i\omega/2}+e^{-i\omega/2}}{2}\right)^{2m}\Bigg|_{\omega=\gamma}\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}\omega}\right)^{2j}\frac{1}{2^{2m}}\sum_{k=0}^{2m}\binom{2m}{k}e^{ik\omega/2}e^{-i(2m-k)\omega/2}\Bigg|_{\omega=\gamma}\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}\omega}\right)^{2j}\frac{1}{2^{2m}}\sum_{k=0}^{2m}\binom{2m}{k}e^{i(k-m)\omega}\Bigg|_{\omega=\gamma}\\ &= \frac{1}{2^{2m}}\sum_{k=0}^{2m}\binom{2m}{k}(-1)^j(k-m)^{2j}e^{i(k-m)\omega}\Bigg|_{\omega=\gamma}\end{aligned}$$

$$= \frac{(-1)^j}{2^{2m}} \left(\binom{2m}{m} \delta(j) + 2 \sum_{r=1}^{m} \binom{2m}{m-r} r^{2j} \cos(r\gamma) \right),$$

where $\delta(j)$ takes values in $\{0,1\}$, and only equals 1 when $j=0$. More straightforwardly, we see that

$$s_{a,m,\ell}^{(2i)}(\gamma) = (-1)^i \sum_{k=0}^{\ell-1} c_k k^{2i} \cos(k\gamma).$$

Using these formulas and a bit of algebra (see below for the case $a=1$), we find that the coefficients c_k satisfy $B^{(a)}c = e_1$, where $B^{(a)}$ is an order-ℓ square matrix with entries $b_{i,k}^{(a)}$, and c is a vector in $\mathbb{R}^\ell$ with entries c_k. The entries of $B^{(a)}$ are given by $b_{0,k}^{(a)} = 1$ for all $0 \le k \le \ell-1, a = 1,2$, and for $1 \le i \le \ell-1$, $0 \le k \le \ell-1$,

$$\begin{aligned} b_{i,k}^{(1)} &= (-1)^i \left[(2k)^{2i} - 2^{-2m} \sum_{r=0}^{2m} \binom{2m}{r} (1 + (-1)^{m+r+k})(k+m-r)^{2i} \right], \\ b_{i,k}^{(2)} &= (-1)^i \left[(2k)^{2i} - 2^{-2m} \sum_{r=0}^{2m} \binom{2m}{r} (k+m-r)^{2i} \right]. \end{aligned} \tag{5.22}$$

We show the calculation for $a=1$, since the other case is similar. We start by plugging the formulas for $f^{(2i)}(\gamma), s_{1,m,\ell}^{(2i)}(\gamma)$ into the left hand side of Equation (5.21):

$$\begin{aligned} &2^{2i}(-1)^i \sum_{j=0}^{\ell-1} c_j j^{2i} - \sum_{k=0}^{i} \binom{2i}{2k} (-1)^k 2^{-2m} \\ &\quad \times \left[\left(\binom{2m}{m} \delta(k) + 2 \sum_{r=1}^{m} \binom{2m}{m-r} r^{2k} \right) \left((-1)^{i-k} \sum_{j=0}^{\ell-1} c_j j^{2(i-k)} \right) \right. \\ &\quad \left. + \left(\binom{2m}{m} \delta(k) + 2 \sum_{r=1}^{m} \binom{2m}{m-r} (-1)^r r^{2k} \right) \left((-1)^{i-k} \sum_{j=0}^{\ell-1} c_j (-1)^j j^{2(i-k)} \right) \right] \\ &= (-1)^i \sum_{j=0}^{\ell-1} c_j \left[(2j)^{2i} - 2^{-2m} \binom{2m}{m} (1 + (-1)^j) j^{2i} \right. \\ &\qquad \left. - 2^{1-2m} \sum_{k=0}^{i} \binom{2i}{2k} \sum_{r=1}^{m} \binom{2m}{m-r} r^{2k} j^{2(i-k)} (1 + (-1)^{r+j}) \right] \\ &= (-1)^i \sum_{j=0}^{\ell-1} c_j \left[(2j)^{2i} - 2^{-2m} \binom{2m}{m} (1 + (-1)^j) j^{2i} \right. \\ &\qquad \left. - 2^{-2m} \sum_{r=1}^{m} \binom{2m}{m-r} (1 + (-1)^{r+j}) ((r+j)^{2i} + (r-j)^{2i}) \right]. \end{aligned}$$

		$m=1$	$m=2$
$\ell=2$	$a=1$	1	$2,1$
	$a=2$	$5,1$	$2,1$
$\ell=3$	$a=1$	1	$2,1$
	$a=2$	$97,24,-1$	$237,124,-1$
$\ell=4$	$a=1$	1	$2,1$
	$a=2$	$24134,6513,-438,31$	$4927,267,-51,5$

		$m=3$	$m=4$
$\ell=3$	$a=1$	33,26,1	29,28,3
	$a=2$	33,26,1	29,28,3
$\ell=4$	$a=1$	33,26,1	1208,1191,120,1
	$a=2$	8306,6567,246,1	1208,1191,120,1
$\ell=5$	$a=1$	33,26,1	1208,1191,120,1
	$a=2$	331505,263856,9252,208,-21	289885,285896,28772,248,-1

Table 5.1: Coefficients for the trigonometric polynomials $s_{a,m,\ell}$. For many pairs of (m,ℓ) here, the actual order of vanishing exceeds the given 2ℓ. We list the coefficients without normalization, in the order $1, \cos(\omega), \cos(2\omega), \ldots$

Now, using the substitutions $r' = m - r$, $r'' = m + r$, and the equation $\binom{2m}{m-r} = \binom{2m}{m+r}$, we see that this equals

$$(-1)^i \sum_{j=0}^{\ell-1} c_j \left[(2j)^{2i} - 2^{-2m}\binom{2m}{m}(1+(-1)^j)j^{2i} \right.$$
$$-2^{-2m}\sum_{r'=0}^{m-1}\binom{2m}{r'}(1+(-1)^{m-r'+j})(m-r'+j)^{2i}$$
$$\left. -2^{-2m}\sum_{r''=m+1}^{2m}\binom{2m}{r''}(1+(-1)^{r''-m+j})(r''-m-j)^{2i} \right],$$

The formula (5.22) for $b_{i,k}^{(1)}$ now follows immediately for all $i > 1$, and the case $i = 0$ just imposes $\sum_{k=0}^{\ell-1} c_k = 1$, which is equivalent to the condition $s_{a,m,\ell}(0) = 1$.

The equations (5.20) give necessary conditions for the order of vanishing at 0 to be correct, but they are not sufficient on their own for (5.15), since the nonnegativity conditions must still be verified. We provide a table of the first several trigonometric polynomials $s_{1,m,\ell}$ and $s_{2,m,\ell}$ in Table 5.1, where all of the conditions of (5.15) hold. $\square$

As an application of Example 5.3, we extend Example 5.2 to arbitrary dimension.

Example 5.4 (Piecewise Linear Box Spline, $n \geq 2$)**.** Let $\Xi = [I\ e]$, where I is the $n \times n$ identity matrix, and e is the column vector of all ones, of length n. Then $d = n+1$, and for

$s_k(\omega) = 1, 1 \leq k \leq n$ and $s_{n+1}(\omega) = (5+\cos(\omega))/6$ (c.f. Table 5.1 entries $(a, m, \ell) = (1, 1, 2)$ and $(2, 1, 2)$), we get $S(\omega) = 1/s_{n+1}(\omega \cdot e)$, and the assumptions (5.15) hold with $\ell = 2$, which is the accuracy and flatness number of $\tau = \left(\prod_{j=1}^{n}(2^{-1}(1+e^{i\omega_j}))\right)(2^{-1}(1+e^{i\omega\cdot e}))$. In this case, $\mu = 1$, which will end up being the number of vanishing moments in the highpass masks we construct.

We obtain an sos representation for $1/S(2\omega) - \sum_{\gamma\in\Gamma^*} |\tau(\omega+\gamma)|^2/S(\omega+\gamma)$ as the sum of (5.18) and (5.19) above, and using the discussion in that example, we can ensure that these sos generators are $\mathcal{G}$-invariant. Then the construction in Theorem 5.1 gives highpass masks satisfying the OEP conditions with τ for this S, and all of the highpass masks have at least 1 vanishing moment. We follow the general calculation for (5.18) and (5.19) in the case $n = 2$ as a demonstration. In this case, we have the same τ as in Example 5.2.

We see that $s_{n+1}(2\omega) - \cos^2(\omega/2)s_{n+1}(\omega) = \sin^4(\omega/2)$, so since $S(\omega) = 6/(5+\cos(\omega\cdot e))$, we have:

$$
\begin{aligned}
&\frac{1}{S(2\omega)} - \sum_{\gamma\in\Gamma^*} \frac{|\tau(\omega+\gamma)|^2}{S(\omega+\gamma)}\\
&= \frac{5+\cos(2\omega\cdot e)}{6} - \sum_{\gamma\in\Gamma^*}\left(\prod_{j=1}^{2}\cos^2((\omega_j+\gamma_j)/2)\right)\frac{\cos^2((\omega+\gamma)\cdot e/2)(5+\cos((\omega+\gamma)\cdot e)}{6}\\
&= \frac{5+\cos(2\omega\cdot e)}{6}\left(1 - \sum_{\gamma_1\in\{0,\pi\}}\cos^2((\omega_1+\gamma_1)/2)\right)\\
&+ \frac{5+\cos(2\omega\cdot e)}{6}\left(\sum_{\gamma_1\in\{0,\pi\}}\cos^2((\omega_1+\gamma_1)/2)\right)\left(1 - \sum_{\gamma_2\in\{0,\pi\}}\cos^2((\omega_2+\gamma_2)/2)\right)\\
&+ \sum_{\gamma\in\Gamma^*}\left(\prod_{j=1}^{2}\cos^2((\omega_j+\gamma_j)/2)\right)\left(\frac{5+\cos(2\omega\cdot e)}{6} - \cos^2((\omega+\gamma)\cdot e/2)\frac{5+\cos((\omega+\gamma)\cdot e)}{6}\right)\\
&= \frac{5+\cos(2\omega\cdot e)}{6}(0) + \frac{5+\cos(2\omega\cdot e)}{6}(1)(0)\\
&+ \sum_{\gamma\in\Gamma^*}\cos^2((\omega_1+\gamma_1)/2)\cos^2((\omega_2+\gamma_2)/2)\sin^4((\omega+\gamma)\cdot e/2),
\end{aligned}
$$

which gives an sos representation for $f(S, \tau; \cdot)$. Comparing with Example 5.2, when $a =$

$5/6$, $b = 0$, $c = 1/6$, we have $f(S,\tau;\cdot) = 1/4 - (1/8)(\cos(2\omega_1) + \cos(2\omega_2))$, which equals

$$\sum_{\gamma\in\Gamma^*} \cos^2((\omega_1+\gamma_1)/2)\cos^2((\omega_2+\gamma_2)/2)\sin^4((\omega+\gamma)\cdot e/2),$$

so the calculations here match those of Example 5.2. Now, if we denote by $g(\omega) = \cos(\omega_1/2)\cos(\omega_2/2)\sin^2(\omega\cdot e/2)$, then applying Lemma 1.2(a), $\sum_\gamma |g^\gamma|^2 = \sum_\nu |g_\nu(2\cdot)|^2$, so we get an sos representation for $f(S,\tau;\cdot)$ as

$$\begin{aligned}&\left|\frac{1}{8}(1-e^{-2i(\omega_1+\omega_2)})\right|^2 + \left|\frac{1}{8}(-1+2e^{-2i\omega_1} - e^{-2i(\omega_1+\omega_2)})\right|^2\\ &+\left|\frac{1}{8}(-1+e^{-2i(\omega_1+\omega_2)})\right|^2 + \left|\frac{1}{8}(-1+2e^{-2i\omega_2} - e^{-2i(\omega_1+\omega_2)})\right|^2.\end{aligned}$$

We can actually combine the first squares in each line and the other two squares to obtain

$$f(S,\tau;\omega) = \left|\frac{1}{4\sqrt{2}}(1-e^{-2i(\omega_1+\omega_2)})\right|^2 + \left|\frac{1}{4\sqrt{2}}(2-e^{2i\omega_1} - e^{-2i\omega_2})\right|^2,$$

which is obviously $O(|\omega|^2)$ for $\omega \approx 0$.

Using the formulas from the proof of Theorem 5.1, we obtain highpass masks (with $\nu \in \{0,1\}^2$ for the $q_{2,\nu}$)

$$\begin{aligned}q_{1,1}(\omega) &= \frac{6}{5+\cos(2\omega\cdot e)}\tau(\omega)(1-e^{2i(\omega_1+\omega_2)})/4\sqrt{2},\\ q_{1,2}(\omega) &= \frac{6}{5+\cos(2\omega\cdot e)}\tau(\omega)(2-e^{-2i\omega_1} - e^{2i\omega_2})/4\sqrt{2},\\ q_{2,\nu}(\omega) &= \frac{1/2}{5+\cos(\omega\cdot e)}e^{i\omega\cdot\nu}((3+\sqrt{6}) + (3-\sqrt{6})e^{i\omega\cdot e})\\ &\quad - \frac{1/2}{5+\cos(2\omega\cdot e)}\tau(\omega)\sum_{\gamma\in\Gamma^*}\overline{\tau(\omega+\gamma)}e^{i(\omega+\gamma)\cdot\nu}((3+\sqrt{6}) + (3-\sqrt{6})e^{i(\omega+\gamma)\cdot e}),\end{aligned}$$

where for the latter group, we used the proof of Lemma 5.1 and read off the top row of $A(\omega)$ to obtain $a_\nu(\omega) = 2^{-1}e^{i\omega\cdot\nu}\frac{1}{2\sqrt{3}}((\sqrt{3}+\sqrt{2}) + (\sqrt{3}-\sqrt{2})e^{i\omega\cdot e})$ (since $S(\omega)^{-1} = \frac{1}{12}|(\sqrt{3}+\sqrt{2}) + (\sqrt{3}-\sqrt{2})e^{i\omega\cdot e}|^2$). □

In the next two examples, we do not give as many details as in the last one, since the ideas are similar.

Example 5.5 (2 Vanishing Moments)**.** Let $\Xi = [I\ I\ e]$, which yields a lowpass mask τ having accuracy number 3. Using Table 5.1 entries $(a, m, \ell) = (1, 2, 2)$ and $(2, 1, 2)$, we get $S(\omega) = \left[\left(\prod_{j=1}^n (2 + \cos(\omega_j))/3\right)(5 + \cos(\omega \cdot e))/6\right]^{-1}$. Since

$$(2 + \cos(2\omega))/3 - \cos^4(\omega/2)(2 + \cos(\omega))/3 - \sin^4(\omega/2)(2 - \cos(\omega))/3 = 0$$
$$(5 + \cos(2\omega))/6 - \cos^2(\omega/2)(5 + \cos(\omega))/6 = \sin^4(\omega/2),$$

the computation in Example 5.3 yields

$$f(S, \tau; \omega) = \sum_{\gamma \in \{0,\pi\}^n} \left(\prod_{j=1}^n \cos^4((\omega_j + \gamma_j)/2)(2 + \cos(\omega_j + \gamma_j))/3\right) \sin^4((\omega + \gamma) \cdot e/2).$$

When $\gamma = 0$, we see that $\sin^4(\omega \cdot e/2)$ has 4 vanishing moments, and when $\gamma \neq 0$, some $\gamma_j = \pi$, in which case $\cos^4((\omega_j + \gamma_j)/2) = \sin^4(\omega_j/2)$ has 4 vanishing moments. Letting $g(\omega) = \left(\prod_{j=1}^n \cos^2(\omega_j/2)((1 + \sqrt{3}) + (-1 + \sqrt{3})e^{i\omega_j})/(2\sqrt{3})\right) \sin^2(\omega \cdot e/2)$, we see that

$$f(S, \tau; \omega) = \sum_{\gamma \in \{0,\pi\}^n} |g^\gamma(\omega)|^2 = \sum_{\nu \in \{0,1\}^n} |(g_\nu)(2\omega)|^2,$$

which is a $\mathcal{G}$-invariant sos representation with 2^n generators. This leads to a collection of 2^{n+1} highpass masks satisfying the OEP conditions with this S and τ, using the method of proof in Theorem 5.1, and all of these highpass masks have at least 2 vanishing moments, by Proposition 5.1.

Example 5.6 (3 Vanishing Moments)**.** Let $\Xi = [I\ I\ I\ e]$, which yields a lowpass mask τ having accuracy number 4. Using Table 5.1 entries $(a, m, \ell) = (1, 3, 3)$ and $(2, 1, 3)$, we get

$$S(\omega) = \left[\left(\prod_{j=1}^n (33 + 26\cos(\omega_j) + \cos(2\omega_j))/60\right)(97 + 24\cos(\omega \cdot e) - \cos(2\omega \cdot e))/120\right]^{-1}.$$

Since

$$s_{1,3,3}(2\omega) - \cos^6(\omega/2)s_{1,3,3}(\omega) - \sin^6(\omega/2)s_{1,3,3}(\omega+\pi) = 0$$
$$s_{2,1,3}(2\omega) - \cos^2(\omega/2)s_{2,1,3}(\omega) = \sin^6(\omega/2)(23 + 8\cos(\omega))/15,$$

calling the right hand side of the second line $r(\omega)$, the computation in Example 5.3 yields

$$f(S,\tau;\omega) = \sum_{\gamma\in\{0,\pi\}^n} \left(\prod_{j=1}^{n} \cos^6((\omega_j+\gamma_j)/2)s_{1,3,3}(\omega_j+\gamma_j) \right) r((\omega+\gamma)\cdot e).$$

When $\gamma = 0$, $\sin^6(\omega \cdot e/2)$ has 6 vanishing moments, and when $\gamma_j = \pi$, $\cos^6((\omega_j+\gamma_j)/2) = \sin^6(\omega_j/2)$ has 6 vanishing moments, so $f(S,\tau;\cdot)$ has 6 vanishing moments. As in the previous examples, we may define g using the relation $f(S,\tau;\cdot) = \sum_\gamma |g^\gamma|^2$, and using the polyphase components of g, we obtain a $\mathcal{G}$-invariant sos representation for $f(S,\tau;\cdot)$. This may be used to obtain the highpass masks satisfying the OEP conditions with S and τ, using the method of proof in Theorem 5.1, and all of these have at least 3 vanishing moments, by Proposition 5.1.

The following example leaves the setting of dyadic dilation, and shows how a tight wavelet frame with maximum vanishing moments may be obtained for the lowpass mask associated with the cubic B-spline refinable function in one dimension with dilation factor 3. In this example, S and τ satisfy the oblique QMF condition.

Example 5.7 (Cubic B-spline with Dilation 3)**.** Consider the lowpass mask for the univariate cubic B-spline with dilation 3, $\tau(\omega) = \left(\frac{1}{3}(1+2\cos(\omega))\right)^3$, which has accuracy number 3 and flatness number 1. Let $S(\omega) = 120/(66 + 52\cos(\omega) + 2\cos(2\omega))$. Then

$$\frac{1}{S(3\omega)} - \sum_{\gamma\in\{0,2\pi/3,4\pi/3\}} \frac{|\tau(\omega+\gamma)|^2}{S(\omega+\gamma)} = 0,$$

and since $1/S(\omega) = |p(\omega)|^2$ with

$$p(\omega) = \frac{1}{4}\left(2(1-2/\sqrt{30}) + 2(1+2/\sqrt{30})\cos(\omega) - 2i\sqrt{1+4/\sqrt{30}}\sin(\omega)\right),$$

we get a tight wavelet frame generated by the masks, for $\nu \in \{0, 1, 2\}$,

$$q_{2,\nu}(\omega) = 3^{-1/2} e^{i\nu\cdot\omega} p(\omega) S(\omega) - 3^{-1/2} S(3\omega)\tau(\omega) \sum_{\gamma \in \{0, 2\pi/3, 4\pi/3\}} \overline{\tau(\omega+\gamma)} e^{i\nu\cdot(\omega+\gamma)} p(\omega+\gamma).$$

Moreover, all three of these highpass masks have $a = 3$ vanishing moments, by Proposition 5.1, since $f(S, \tau; \cdot)$ has $m = \infty$ vanishing moments. $\square$

5.4 Summary

In this chapter, we applied the ideas of Chapter 4 to prove that the oblique sub-QMF condition is equivalent to the existence of highpass masks satisfying the oblique extension principle conditions with a given lowpass mask and vanishing moment recovery function. In particular, we used Theorem 4.1 to obtain sors representations for certain nonnegative trigonometric polynomials, and we used these sors generators to construct the desired highpass masks. We also proved a proposition about the number of vanishing moments of the highpass masks in our construction (see Proposition 5.1). On the analytical side, we used the oblique sub-QMF condition to prove that the constructed highpass masks generate a tight wavelet frame, and showed that even when S is not a rational trigonometric polynomial, the OEP conditions necessitate that the oblique sub-QMF condition holds on a large set, which is enough to show that the highpass masks still generate a tight wavelet frame (see Section 5.1.2).

In the examples, we focused on the case of box splines with dyadic dilation, and showed how to find a vmr function which is a simple product of univariate trigonometric polynomials, but still satisfies the oblique sub-QMF condition with the box spline lowpass mask τ. Moreover, we showed that this may be used to construct a tight wavelet frame with highpass masks having vanishing moments equal to the accuracy number of the "separable part" τ_0 of τ, if we think of $\tau = \tau_0\tau_1$, where τ_0 is a tensor product box spline, and τ_1 is the product of the remaining factors. While it is possible to use the methods of Theorem 5.1 to construct tight wavelet frames with maximum vanishing moments when given the appropriate S, finding vmr functions for which $f(S, \tau; \cdot)$ has the appropriate number of vanishing moments may be complicated (though some methods are known, like the one in [42]). As such, one way of using our theorem is to get better flexibility in construction with the OEP: in Examples 5.5 and 5.6, we traded one vanishing moment for a much simpler form of S, which we were able to describe for the box spline having that form in any dimension.

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